Universidade Federal de Pernambuco

Departamento de Física

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Primeiro semestre de 2019

Quantum Mechanics

27/02/2019 - 09:00 to 12:00

(Choose three out of the four proposed questions)

Question 1: Space of states

Consider the orthonormal states $|1\rangle$, $|2\rangle$, $|3\rangle$ in a given Hilbert space. It is known that

i) the observable $\hat{\xi}$ has $|1\rangle$ and $|2\rangle$ as eigenvectors, both corresponding to the eigenvalue ξ_1, and the eigenvector $|3\rangle$ with eigenvalue ξ_3;

ii) the observable $\hat{\eta}$ has eigenvectors $|1\rangle$, with eigenvalue η_1, and $|2\rangle$ and $|3\rangle$, with eigenvalue η_2.

Questions:

a) (20%) The observables $\hat{\xi}$ and $\hat{\eta}$ are compatible? Explain your answer. Does the set $|1\rangle$, $|2\rangle$, $|3\rangle$ form a complete base in the description of these observables?

   Consider now that the system was prepared in a state given by $|A\rangle = (a\,|1\rangle + b\,|2\rangle + c\,|3\rangle)\,/\sqrt{a^2+b^2+c^2}$ (where a, b and c are real numbers), assume that the observable $\hat{\xi}$ is measured first, and then $\hat{\eta}$.

b) (20%) What is the state of the system immediately after the first measurement resulted in the value ξ_1?

c) (20%) What is the probability that the first measurement results in ξ_1?

d) (20%) If the first measurement results in a value ξ_1, what is the probability that the second measurement results in the value η_1?

e) (20%) If the first measurement results in the value ξ_3, what is the probability that the second measurement results in the value η_2?

Question 2: Infinite well potential

As part of a project of building a solid state laser, consider an electron placed in an one dimensional rectangular well potential of width L, for which $V(x) = 0$ if $0 < x < L$ and $V(x) = \infty$ in all other regions.

a) (15%) Find an expression for the wave length of the electron in the ground state as function of the length L. Repeat the calculation for the electron in the first excited state. Sketch the corresponding wave functions.

b) (15%) Find an expression for the wave length of the emitted photon λ_{photon} in the transition from the first excited state to the ground state in terms of L.

c) (25%) Find an expression for the matrix element $|\langle\varphi_m|\,\hat{x}\,|\varphi_n\rangle|$ in terms of L, corresponding to the transition from the first excited state to the ground state, with $|\varphi_i\rangle$ representing the quantum state i.

d) (10%) The lifetime for a spontaneous emission between states $|\varphi_n\rangle$ and $|\varphi_m\rangle$ can be written as $\tau_{esp}^{nm} = (A_{nm})^{-1}$, where $A_{nm} = (7,2 \times 10^{17}/\lambda_{photon}^3)\,|\langle\varphi_m|\,\hat{x}\,|\varphi_n\rangle|^2$ (with all lengths in nm) is the Einstein coefficient for the spontaneous emission. For $L = 12$ nm, find the lifetime for the spontaneous emission from the first excited state.

Consider now that the system is initially prepared with the electron in the ground state $|\varphi_1\rangle$, with eigenenergy E_1. Then, in time $t = 0$, the potential is rapidly modified, in a manner that the original wave function remains the same, but the potential now is $V(x) = 0$ for $0 < x < 2L$ and $V(x) = \infty$ for all other regions.

e) (35%) Determine the probability that a measurement in time $t > 0$ reveals that the new system is in the first excited state.

Data:

$$h = 6,63 \times 10^{-34}\ \text{J.s} = 4,14 \times 10^{-15}\ \text{eV.s} \quad \hbar = 1,05 \times 10^{-34}\ \text{J.s} = 6,58 \times 10^{-16}\ \text{eV.s}$$

$$m_e = 9,11 \times 10^{-31}\ \text{kg} \quad c = 3,0 \times 10^{8}\ \text{m/s}$$

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \quad \sin(a+b) = \sin a \cos b + \cos a \sin b$$

$$\int_0^L x \cos\left(\frac{m\pi x}{L}\right) dx = -\frac{2L^2}{m^2\pi^2}\ (m\ \text{odd})$$

$$\int_0^\pi \sin(mx)\sin(nx)dx = \begin{cases} 0 & \text{if}\quad m \neq n \\ \pi/2 & \text{if}\quad m = n \end{cases} \quad \text{for } m \text{ and } n \text{ positive integers}$$

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$$\int_0^\pi \sin(mx)\cos(nx)dx = \begin{cases} 0 & \text{if}\quad m+n \quad \text{odd} \\ \frac{2m}{m^2-n^2} & \text{if}\quad m+n \quad \text{even} \end{cases} \quad \text{for } m \text{ and } n \text{ integers}$$

Question 3: Time evolution

An electron in the presence of an uniform magnetic field of magnitude B_0 pointing in the direction of positive z axis is found at time $t = 0$ in the eigenstate of the spin operator $\hat{S}_y$ with eigenvalue $-\hbar/2$. The Hamiltonian of the system can be written as $\hat{H} = -\gamma\vec{S}\cdot\vec{B}$, with γ a parameter of the system.

a) (10%) Obtain the time evolution operator $\hat{U}(t)$ for the system.

b) (30%) Find the initial state of the system in the basis that diagonalizes $\hat{U}(t)$.

c) (20%) Obtain the wave functiona $|\Psi(t)\rangle$ at time t.

d) (40%) What is the probability that a measurement of $\hat{S}_x$ will return the value $\hbar/2$ in time t?

Data:

$$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; ; \; \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; ; \; \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Question 4: Harmonic Oscillator

Consider a bi-dimensional harmonic oscillator described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2) \, .$$

a) (20%) From the annihilation operators given by

$$\hat{a}_x = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i\hat{p}_x}{m\omega} \right)$$

and their corresponding creation operators, show that the Hamiltonian can be diagonalized in the excitation number basis.

b) (20%) Determine the energies of the system and discuss the possible degeneracies of the corresponding eigenstates.

c) (20%) Assume now that the oscillations are coupled by the perturbation potential $\hat{V} = \lambda \hat{x}\hat{y}$. Write $\hat{V}$ in terms of the creation and annihilation operators for the basis of unperturbed states. What is the result of the action of $\hat{V}$ over one of the first excited unperturbed states?

d) (40%) Show that the perturbation defined above breaks the degeneracy of the first excited state. Determine the energies of this state.