Federal University of Pernambuco

Department of Physics

Graduate Program

Qualifying Exam 2018

Second Semester

Quantum Mechanics

August 9, 2018 - From 09:00 to 12:00

(Choose three among the four problems)

PROBLEM 1: SIMPLE APLICATIONS AND TIME INDEPENDENT POTENTIALS

Consider a particle of mass m submitted to a time independent potential $V = \dfrac{m\omega^2 x^2}{2}$, where x is the position and ω the angular frequency of the oscilator.

(a) (20%) Obtain the wave function corresponding to the first excited state at $t = 0$, $\Psi_1(x,\, 0)$.

Now, suppose the quantum state of the particle is described by the wave function

$$\Psi(x,\, 0) = C\left[\Psi_0(x) + 3\Psi_1(x)\right],$$

where $\Psi_0(x)$ is the wave function corresponding to the ground state.

(b) (30%) Calculate the constant C and the probability to find the particle in a region of width $\mathrm{d}x$ around the position x.

(c) (30%) Determine both the amplitude and the oscillation frequency of $\langle x \rangle$.

(d) (20%) When one measures the energy of the particle, which values can be obtained and what are the probabilities to find them?

Formula list:

$$\Psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega x^2}{2\hbar}\right)$$

$$a_{\pm} = \sqrt{\frac{m\omega}{2\hbar}}\left(x \pm \frac{i\,p}{m\omega}\right)$$

$$\int_0^{\infty} \mathrm{d}x\, x^2\, e^{-x^2} = \frac{\sqrt{\pi}}{4}$$

Problem 2: Angular momentum

Consider the position and momentum operators along two orthogonal directions for two one-dimensional harmonic oscillators, which satisfy the commutation relations $[x, p_x] = i\hbar = [y, p_y]$, $[x, y] = [x, p_y] = [y, p_x] = [p_x, p_y] = 0$.

(a) (30%) Find the commutation relations between the operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip_x}{m\omega}\right), \quad a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip_x}{m\omega}\right),$$

$$b = \sqrt{\frac{m\omega}{2\hbar}}\left(y + \frac{ip_y}{m\omega}\right), \quad b^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(y - \frac{ip_y}{m\omega}\right).$$

(b) (20%) Obtain $xp_y - yp_x$ in terms of the operators a, $a^\dagger$, b e $b^\dagger$.

(c) (30%) Define $J_+ = \hbar\, a^\dagger b$, $J_- = \hbar\, b^\dagger a$, and $J_z = \frac{\hbar}{2}\left(a^\dagger a - b^\dagger b\right)$. Show that these operators satisfy the following commutation relations

$$[J_z, J_\pm] = \pm\hbar J_\pm, [J_+, J_-] = 2\hbar J_z.$$

(d) (20%) Show that $xp_y - yp_x$ commutes with the total number of excitations of the oscillators $N = a^\dagger a + b^\dagger b$.

Formula list:

$$[AB, CD] = A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B$$

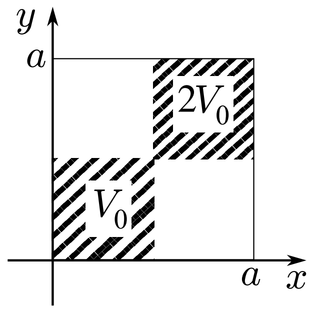
PROBLEM 3: PERTUBATION THEORY

Consider a particle of mass m confined to the infinite two-dimensional well:

$$V(x,\, y) = \begin{cases} 0, & \text{if } 0 < x < a \text{ and } 0 < y < a, \\ \infty, & \text{otherwise.} \end{cases}$$

Now, the following pertubation is added to the system

$$\mathcal{H}' = \begin{cases} V_0, & \text{if } 0 < x < a/2 \text{ and } 0 < y < a/2, \\ 2V_0, & \text{if } a/2 < x < a \text{ and } a/2 < y < a, \\ 0, & \text{otherwise.} \end{cases}$$



(a) (30%) Calculate the first order correction to the energy of the ground state.

(b) (40%) Calculate the first order correction to the energy of the first excited state.

(c) (30%) Obtain the normalized wave function of the perturbed first excited state.

Formula list:

$$\int \operatorname{sen}^2 x \; \mathrm{d}x = \frac{x - \operatorname{sen} x \cos x}{2} + \text{constant},$$

$$\int \operatorname{sen}(ax)\operatorname{sen}(bx)\; \mathrm{d}x = \frac{b \operatorname{sen}(ax)\cos(bx) - a\cos(ax)\operatorname{sen}(bx)}{a^2 - b^2} + \text{constant}, \qquad \text{if } a \neq b$$

Problem 4: Identical particles

The two electron state in a hellium atom is described by

$$|r_1, r_2, m_1, m_2\rangle = |r_1, r_2\rangle\,|m_1, m_2\rangle\,,$$

where the spatial $|r_1, r_2\rangle$ and the spin $|m_1, m_2\rangle$ dependencies are separable. Each spin $m_i = \pm$, where $i = 1, 2$, may be parallel or antiparallel to a reference axis.

(a) **(50%)** Let P denote the operator which interchange electrons 1 and 2. Find a base in which P is diagonal.

Sugestion: Check how P acts on $|m_1, m_2\rangle$.

(b) **(20%)** Consider that the spatial dependence $|r_1, r_2\rangle$ of a given state is symmetric under the permutation between r_1 and r_2, $P\,|r_1, r_2\rangle = |r_2, r_1\rangle$. Determine the possible spin states of the system.

(c) **(30%)** Consider that the spatial dependence $|r_1, r_2\rangle$ is antisymmetric with respect to the permutation between r_1 e r_2, $P\,|r_1, r_2\rangle = -\,|r_2, r_1\rangle$. Determine the possible spin states of the system.

Formula list:

$$P\,|\psi\rangle = |\psi\rangle \quad \text{(bosons)}, \qquad P\,|\psi\rangle = -\,|\psi\rangle \quad \text{(fermions)},$$

$$S_{iz}\,|-\rangle = -\frac{\hbar}{2}\,|-\rangle\,, \qquad S_{iz}\,|+\rangle = \frac{\hbar}{2}\,|+\rangle$$
