Federal University of Pernambuco

Department of Physics

Graduate Program

Qualifying Exam 2018

Second Semester

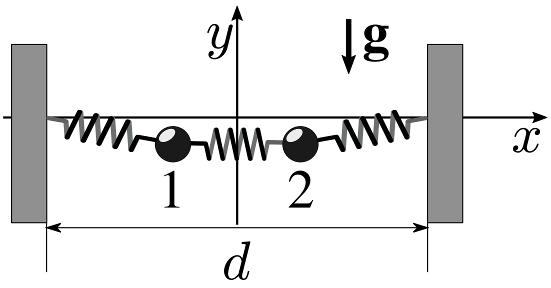
Classical Mechanics

August 10, 2018 - From 09:00 to 12:00

(Choose three among the four problems)

Problem 1: Oscillations

The figure below shows two small spheres of mass m each, which are connected to three springs having the same elastic constant k. Two of the springs are fixed to vertical walls. The points where the springs are attached to the walls are at the same height and separated by a horizontal distance d. For convenience, the origin of the coordinates is placed at a distance $d/2$ and at the same height of the points the spring are fixed. The local gravitational acceleration is $\mathbf{g} = -g\hat{y}$.



(a) (10%) Write the equations of motion for each sphere.

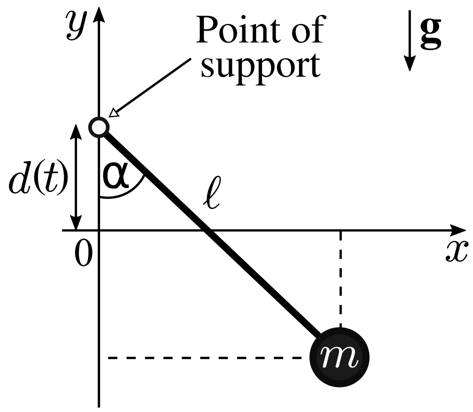
(b) (30%) Determine the equilibrium positions of each sphere.

(c) (40%) Use the transformation of variables $\mathbf{q} = \mathbf{u}_2 - \mathbf{u}_1$ and $\mathbf{Q} = \mathbf{u}_2 + \mathbf{u}_1$ to decouple the equations of motion. Obtain the normal modes of vibration of the system (i.e., describe the collective modes and calculate their oscillation frequencies).

(d) (20%) Now, consider that both spheres are held at rest, with sphere 1 at its equilibrium position [obtained in (b)] and sphere 2 at the origin of the coordinates. At $t = 0$ the system is released. Calculate the time dependence of the positions of spheres 1 and 2 for $t \geq 0$.

Problem 2: Lagragian formalism

A simple pendulum of length ℓ and mass m, placed in an uniform gravitational field, makes an angle α with the vertical y axis, as depicted in the figure. The gravitation acceleration is $\mathbf{g} = -g\hat{y}$, where $\hat{y}$ is the unit vector along the y axis. The point of support of the pendulum moves along the vertical direction with its time dependent position given by $d(t)$. Consider the gravitational potential energy zero at $y = 0$. Disregard friction and air resistance effects.
Attention: Use α as generalized coordinate.



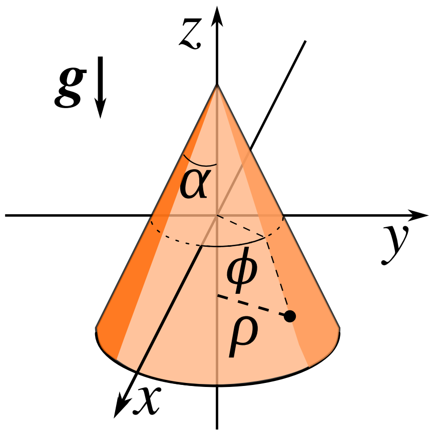
(a) **(20%)** Write the Lagrangian of the system.

(b) **(40%)** Obtain the equation of motion for the angle α by using the Euler-Lagrange equation.

(c) **(40%)** Consider $d(t) = A\,t^2/2$, where A is a positive constant. Calculate the oscillation period of the pendulum for small oscillations (small α values).

Problem 3: Hamiltonian formalism and central forces

A particle of mass $m = 1$ (in arbitrary units) is restricted to move on the surface of a right circular cone with vertex up, whose generatrix makes an angle α with the z axis (see figure). The particle trajectory can be described by the time dependent cylindrical coordinates $(\rho(t),\, \phi(t),\, z(t))$. The local gravitational acceleration is $\mathbf{g} = -g\hat{z}$, where g is constant.



(a) (35%) Obtain the Hamiltonian $\mathcal{H}$ of the system in cylindrical coordinates.

(b) (25%) Which quantities are conserved while the particle moves on the cone surface? Justify your answer.

Now, consider that at $t = 0$ the particle is on the x axis, at a distance ρ_0 from the z axis, and with radial velocity v_ρ satisfying

$$v_\rho^2 + \frac{\ell^2 \mathrm{sen}^2\alpha}{\rho_0^2} = 2g\rho_0 \mathrm{sen}\,\alpha \cos\alpha,$$

where ℓ is the angular momentum z component.

(c) (25%) By substituting $u = \ell/\rho$ in the mechanical energy of the system, show that

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\phi}\right)^2 + \mathrm{sen}^2\alpha\, u^2 = \frac{2\ell g\, \mathrm{sen}\,\alpha \cos\alpha}{u}.$$

(d) (15%) Determine the particle trajectory, i.e., $\rho = \rho(\phi)$, for $\ell \neq 0$.

Sugestion: Use $s^2 = \dfrac{\tan\alpha}{2\ell g} u^3$ and solve the resulting integral.

Formula list:

$$\int \frac{\mathrm{d}w}{\sqrt{1 - w^2}} = \arcsin w + \text{constant}$$

PROBLEM 4: CANONICAL TRANSFORMATIONS

Consider the following transformation of variables from (q, p) to (Q, P):

$$Q = p + iaq, \qquad P = \frac{(p - iaq)}{2ia},$$

where a is a constant.

(a) (20%) Is this a canonical transformation? Justify your answer quantitavely in detail.

(b) (45%) Find the generating function $S = S(q, Q)$ for this transformation.

(c) (35%) The Hamiltonian of the system (q, p) is given by $\mathcal{H} = (p^2 + a^2q^2)/2$. Write the Hamilton equations for the (Q, P) coordinates. Obtain $Q(t)$ and $P(t)$ using the initial conditions $Q(0) = Q_0$ and $P(0) = P_0$ at $t = 0$.
