UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS EXATAS E DA NATUREZA
PROGRAMA DE PÓS-GRADUAÇÃO EM ESTATÍSTICA

CLAUDIO JAVIER TABLADA

GENERALIZED PROBABILITY DISTRIBUTIONS FOR LIFETIME APPLICATIONS

Recife
2017

CLAUDIO JAVIER TABLADA

GENERALIZED PROBABILITY DISTRIBUTIONS FOR LIFETIME APPLICATIONS

Doctoral thesis submitted to the Graduate Program in Statistics, Department of Statistics, Federal University of Pernambuco as a partial requirement for obtaining a Ph.D. in Statistics.

Advisor: Professor Dr. Gauss Moutinho Cordeiro

Recife
2017

Catalogação na fonte
Bibliotecária Monick Raquel Silvestre da S. Portes, CRB4-1217

T113g Tablada, Claudio Javier
Generalized probability distributions for lifetime applications / Claudio Javier Tablada. – 2017.
109 f.: il., fig., tab.

Orientador: Gauss Moutinho Cordeiro.
Tese (Doutorado) – Universidade Federal de Pernambuco. CCEN, Estatística, Recife, 2017.
Inclui referências.

1. Probabilidade. 2. Teoria das distribuições. I. Cordeiro, Gauss Moutinho (orientador). II. Título.

519.2 CDD (23. ed.) UFPE- MEI 2017-42

CLAUDIO JAVIER TABLADA

GENERALIZED PROBABILITY DISTRIBUTIONS FOR LIFETIME APPLICATIONS

Tese apresentada ao Programa de Pós-Graduação em Estatística da Universidade Federal de Pernambuco, como requisito parcial para a obtenção do título de Doutor em Estatística.

Aprovada em: 23 de janeiro de 2017.

BANCA EXAMINADORA

Prof. Gauss Moutinho Cordeiro
UFPE

Prof. Getúlio José Amorim do Amaral
UFPE

Prof.ª Maria do Carmo Soares de Lima
UFPE

Prof. Frank Sinatra Gomes da Silva
UFRPE

Prof. Claudio Tadeu Cristino
UFRPE

A minha esposa Renilma

Sou grato pelo Universo conspirar para que
as sequências de eventos aleatórios de nossas vidas
convergirem certamente para o mesmo ponto.

Agradecimentos

Em fevereiro de 2012, ao chegar neste país para fazer pós-graduação, não poderia imaginar as experiências e os acontecimentos que me esperavam. Aqui, tive o privilégio de conhecer pessoas maravilhosas que me acolheram e me fizeram sentir em casa. Sou grato a minha esposa Renilma, por estar ao meu lado me apoiando, dando sempre uma palavra de conforto nos momentos difíceis e fazer de mim uma pessoa melhor e mais feliz. Agradeço a minha família, principalmente a minha mãe, Leontina Margarita, pela paciência, cuidado, incentivo aos meus estudos e a compreensão pelos momentos em que estive ausente. Gostaria de agradecer a minha família brasileira, principalmente a meus sogros, Raminha e Renivaldo, por sempre me incluírem em suas orações e torcer por mim. Um agradecimento especial ao meu orientador, Prof. Gauss Cordeiro, pela orientação, apoio e confiança. Foi um privilégio trabalhar com um profissional com tanta dedicação e paixão pela ciência e que tenho como um pesquisador de referência na comunidade acadêmica. Gostaria de agradecer a todos os meus amigos e colegas que fizeram parte desta trajetória, especialmente a Pedro Rafael, Giannini Italino e Fernando Ramirez. Agradeço a todos os professores do Departamento de Estatística da UFPE que contribuíram para minha formação acadêmica, especialmente ao Prof. Klaus Vasconcellos, pelas suas aulas inspiradoras e todo o conhecimento compartilhado. Agradeço a meu amigo, Prof. German Torres, por ter incentivado a minha vinda para o Brasil e direcionado minha carreira científica na Argentina. Também agradeço a Valéria Bittencourt, por ser uma pessoa atenciosa e eficiente, colocando-se sempre à disposição para ajudar. Finalmente, gostaria de agradecer à banca examinadora pelas sugestões e contribuições dadas a este trabalho e à CAPES, pelo apoio financeiro.

Abstract

The art of parameter induction to a parent distribution is one of the methods more used for obtain more versatile models. The main reason for this trend is the fact that, many times, classic models often may not be flexible enough to adjust certain lifetime data. So, generalized or extended distributions are of great importance, mainly for two reasons: for controlling the tails and improve the goodness-of-fit of the parent distribution. In this thesis, we propose two new families of distributions, namely the supremum and infimum families, which induce a shape parameter to a parent distribution. We obtain some properties and mathematical quantities of these families. In addition, we present five particular models belonging to the supremum family and others five models belonging to the infimum family. Other contribution is a three-parameter model called the modified Fréchet distribution, which is obtained by inducing a shape parameter in the Fréchet model. Using the Lambert W function, we obtain several mathematical quantities and properties of this model. Finally, we propose a four-parameter generalized model called the beta Marshall-Olkin Lomax distribution, which is obtained to considering the Lomax distribution as the parent model in the beta Marshall-Olkin gerator. We obtain several useful expansions and mathematical properties for this model. In all cases, we prove empirically the applicability of the new models to real data.

Keywords: Generalized models. Lambert W function. Lifetime analysis. Parametric induction.

Resumo

A arte da indução paramétrica a uma distribuição-base é um dos métodos mais usados para obter modelos mais versáteis. A principal razão para esta tendência é o fato de que, muitas vezes, modelos clássicos podem não ser suficientemente flexíveis para ajustar certos dados de tempos de vida. Assim, distribuições generalizadas ou estendidas são de grande importância, principalmente por duas razões: para ter maior controle nas caudas e para melhorar a bondade de ajuste da distribuição-base. Nesta tese, propomos duas novas famílias de distribuições, denominadas de famílias do supremo e do ínfimo, as quais acrescentam um parâmetro de forma a uma distribuição-base. Obtemos algumas propriedades e quantidades matemáticas dessas famílias. Além disso, apresentamos cinco modelos particulares pertencentes à família do supremo e outros cinco modelos pertencentes à família do ínfimo. Uma outra contribuição é um modelo de três parâmetros, denominado de distribuição Fréchet modificada, a qual é obtida acrescentando um parâmetro de forma no modelo Fréchet. Usando a função W de Lambert, obtemos várias quantidades e propriedades matemáticas deste modelo. Finalmente, propomos um modelo generalizado de quatro parâmetros, denominado de distribuição beta Marshall-Olkin Lomax, obtido considerando a distribuição Lomax como modelo base no gerador beta Marshall-Olkin. Determinamos várias expansões úteis e propriedades matemáticas para este modelo. Em todos os casos, provamos empiricamente a aplicabilidade dos novos modelos a dados reais.

Palavras-chave: Análise de tempo de vida. Função W de Lambert. Indução paramétrica. Modelos generalizados.

List of Figures

List of Tables

Contents

Chapter 1

Introduction

Concepts of lifetime describe how a component or system evolve over time. So, lifelengths of devices or of biological organisms are the principal focus in the field of lifetime (survival) analysis. These phenomenons are generally described by nonnegative random variables, which are often assumed to be absolutely continuous having density functions with support in the positive real line. In this setting, distributions with such support play a fundamental role in lifetime applications.

The theory of distributions with support in the positive real line has grown widely in the last years, becoming one of the main statistical tools to model lifetime data. The main reason for this approach is the fact that many basic distributions used in lifetime analysis have a limited range of behaviour and can not represent all situations in real applications. Therefore, at present, there is a great interest in obtaining new and more flexible lifetime models.

Another important reason to introduce new lifetime models is the fact that hazard rate functions (hrf's) of lifetime variables may exhibit various shapes depending on many factors. Some shape properties of the hrf have important implications in practice. Thus, generalized or extended distributions allow to provide more versatile models that present hrf's with the classical shapes: increasing, decreasing, unimodal and bathtub.

The method more used to generate new lifetime models is by inducing one or more additional shape parameters to a parent distribution G in order to add more flexibility. Within this framework, the most known method of parameter induction is that one by compounding existing distributions, usually referred as generalized G families of distributions.

Several generalized distributions have been studied in the literature. Some of the best known are: the Marshall-Olkin extended (MOE) family (MARSHALL; OLKIN, 1997), exponentiated generated (exp-G) family (GUPTA *et al.*, 1998), beta-generated (beta G) family (EUGENE *et al.*, 2002), gamma-generated (gamma G) family (ZO-

GRAFOS; BALAKRISHNAN, 2009; RISTIĆ; BALAKRISHNAN, 2012), Kumaraswamy generalized (Kw G) family (CORDEIRO; DE CASTRO, 2011), transformed-transformer ($T-X$) method (ALZAATREH *et al.*, 2013) and transmutated family of distributions (SHAW; BUCKLEY, 2009; BOURGUIGNON *et al.*, 2016; NOFAL *et al.*, 2016). A detailed compilation of these families is given by Tahir & Nadarajah (2015).

Based on parameter induction, we propose new models primarily to be used in lifetime applications, although they may also be suitable for fit data of another nature. This thesis is organized in five chapters. Chapters 2, 3 and 4 are independent and are presented in scientific manuscripts format, which means that they can be read separately. Thus, some results and notations used are introduced more than once.

In Chapter 2, we define two new families of distributions, named the *supremum* and *infimum* families, by inducing one additional shape parameter to a parent distribution G. The name of these new families comes from an analogy with the mathematical properties of supremum and infimum of a sequence of real numbers. Several special lifetime distributions belonging to the proposed classes are presented and some general properties of these families are provided. Because the supremum and infimum families can also be obtained as distributions of maximum and minimum functions of a sequence of independent and identically distributed (i.i.d.) random variables, they have a direct physical interpretation. In addition, the hrf of the supremum family can be expressed as sums of hrf's of two independent random variables. Because of this, we have the important fact that this family can induce bathtub shape in its hrf. In addition, we consider the estimation of the parameters of these families by the maximum likelihood method. Since a simulation study is fundamental in new distributions, we perform a Monte Carlo simulation experiment in order to evaluate the maximum likelihood estimates (MLE's) of the Fréchet supremum model.

An absolutely continuous model which has wide applicability in extreme value theory is the Fréchet distribution. However, this model has a limited range of behaviour and can not represent all the situations found in applications. For example, this model does not allow decreasing or inverted unimodal hazard rate, which is widely used in lifetime applications by its considerable intuitive appeal. Following a similar approach to that given in Lai *et al.* (2003) to define the modified Weibull model, it is introduced in Chapter 3 the three-parameter *modified Fréchet* distribution by extending the Fréchet distribution. Differently from the Fréchet model, the new distribution allows decreasing and inverted unimodal hazard rate, which is useful for modeling various cases in lifetime applications (LAI, 2013; MARSHALL; OLKIN, 1997). Several mathematical quantities and properties of the new distribution are obtained by considering the Lambert W function (CORLESS *et al.*, 1996; JODRÁ, 2010), which has been applied commonly to solve problems formulated in terms of logarithmic or exponential equations. We consider the estimation of the parameters of the new model by the maximum likelihood method. In addition, we perform a Monte Carlo simulation experiment in order to evaluate the MLE's.

Chapter 4 introduces the four-parameter *beta Marshall-Olkin Lomax* distribution, which is obtained by considering the Lomax distribution as the parent model in the generated beta Marshall-Olkin family proposed by Alizadeh *et al.* (2015). Given that the density function of the new distribution can be expressed as a linear combination of Lomax and exponentiated Lomax densities, several properties of the new model can be easily derived from the properties of those latter. In addition, we present explicit expressions for some statistical quantities. We consider the estimation of the parameters of the BMOL model by the maximum likelihood method and perform a Monte Carlo simulation experiment in order to evaluate such estimates. Finally, in Chapter 5, we offer the final conclusions and outline some future research lines.

Besides obtaining the mathematical quantities and properties of all distributions introduced, we prove empirically in each chapter the potenciality of the new models by means of applications to real data sets. In other words, they are very appropriate for lifetime applications.

Chapter 2

The supremum and infimum families of distributions

Resumo

Neste capítulo, definimos duas novas famílias de distribuições G-generalizadas ao introduzir um parâmetro de forma a uma distribuição-base G. Essas novas famílias, denotadas por G^{sup} e G^{inf}, têm uma interpretação física direta e expressões simples para os momentos em termos de momentos de distribuições exp-G. Provamos que a família G^{sup} pode apresentar forma de banheira na sua função razão de risco, além de ambas famílias proporcionar maior flexibilidade. Várias distribuições particulares pertencentes às famílias propostas são dadas. Consideramos a estimação dos parâmetros destas famílias pelo método de máxima verossimilhança e realizamos uma simulação de Monte Carlo com o objetivo de avaliar essas estimativas no modelo Fréchet supremo. Provamos empiricamente a utilidade das novas famílias por meio de duas aplicações a dados reais.

Palavras-chave: Análise de tempo de vida, famílias G-generalizadas, indução paramétrica.

Abstract

In this chapter, we introduce two new generalized G families by inducing one additional shape parameter to a parent distribution G. These new families, named G^{sup} and G^{inf}, have a direct physical interpretation and simple expressions for moments in terms of moments of exp-G distributions. We prove that the G^{sup} family can induce bathtub shape in its hazard rate function, besides both families provide greater flexibility. Several particular lifetime distributions belonging to the proposed families are given. We consider the estimation of the parameters of these families by the maximum likelihood method and perform a Monte Carlo simulation experiment in order to evaluate these estimates in the Fréchet supremum model. The potentiality of the new families is proved empirically by means of two applications to real data sets.

Keywords: Generalized G families, lifetime analysis, parametric induction.

2.1 Introduction

Parametric induction of one or more additional shape parameters to a parent distribution G is one of the most useful methods for obtaining new and more flexible families, mainly for use in lifetime applications. This approach has proved to be useful in two main situations: (i) to make the generated distribution more flexible for studying the tail properties and (ii) to improve the goodness-of-fit of the proposed generalized family of distributions (TAHIR; NADARAJAH, 2015). Thus, many families of distributions have been proposed in the literature, such as the exponentiated-generated, beta-generated (EUGENE *et al.*, 2002), gamma-generated (ZOGRAFOS; BALAKRISHNAN, 2009), Kumaraswamy-generated (CORDEIRO; DE CASTRO, 2011) and transmuted families (SHAW; BUCKLEY, 2009; BOURGUIGNON *et al.*, 2016; NOFAL *et al.*, 2016), among others.

Although these families generalize a parent distribution, often they do not enjoy of a simple physical interpretation or do not provide models that present bathtub hazard rates, which have considerable intuitive appeal in lifetime applications. These properties have important consequences in practice, once they allow to obtain sufficiently flexible models with a wide range of applicability.

Aiming to obtain models with such properties, we propose two new families of distributions, named G^{sup} and G^{inf} families, by inducing one additional shape parameter to a parent distribution G. Since these families can also be obtained as distributions of maximum and minimum functions of a sequence of independent and identically distributed (i.i.d.) random variables, they have a direct physical interpretation. The hazard rate function (hrf) and reverse hazard rate function (rhrf) for the new families are provided and we prove that the hrf of the G^{sup} family can be expressed as sums of hrf's of two independent random variables. Because of this, we have the important fact that the G^{sup} family can induce bathtub shape in its hrf.

The chapter unfolds as follows. In Section 2.2, we introduce the G^{sup} and G^{inf}

families. In Section 2.3, we provide some motivations to validate these families. Section 2.4 gives several special lifetime distributions belonging to the proposed families and we prove how the G^{sup} family can induce bathtub shape in its hrf. Section 2.5 deals with the study of shapes of probability densities functions (pdf's) and hrf's of the supremum and infimum families. In Section 2.6, we obtain some properties of the new families. In Section 2.7, we provide a general method to generate random variates from these families. Section 2.8 is devoted to maximum likelihood estimates (MLE's) for complete samples and, in Section 2.9, we carry out a simulation study to evaluate the performance of these estimates for a special model. In Section 2.10, the potentiality of the proposed families is proved empirically by means of two applications. Finally, Section 2.11 concludes the chapter.

2.2 The supremum and infimum families

Based on the $T-X$ method (ALZAATREH *et al.*, 2013), we propose two new families to inducing one additional shape parameter to an absolutely continuous model.

For $b > 0$, consider the functions

$$W_b^{sup}(z) = z + z^b - z^{b+1}, \quad W_b^{inf}(z) = z[1-(1-z)^b].$$

For $0 \leq z \leq 1$, the functions $W_b^{sup}(z)$ and $W_b^{inf}(z)$ satisfy the following properties:

i) $W_b^{sup}(0) = W_b^{inf}(0) = 0$ and $W_b^{sup}(1) = W_b^{inf}(1) = 1$,

ii) $W_b^{sup}(z)$ and $W_b^{inf}(z)$ are continuously differentiable with derivatives

$$w_b^{sup}(z) = \frac{d}{dz} W_b^{sup}(z) = 1 + b\, z^{b-1} - (b+1)\, z^b,$$
$$w_b^{inf}(z) = \frac{d}{dz} W_b^{inf}(z) = 1 - (1-z)^b + b\, z\, (1-z)^{b-1}.$$

iii) $W_b^{sup}(z)$ and $W_b^{inf}(z)$ are strictly increasing in $[0,1)$, that is, $w_b^{sup}(z)$,$w_b^{inf}(z) > 0$ for $0 \leq z < 1$.

From the properties above, we note that $W_b^{sup}(z)$ and $W_b^{inf}(z)$ are the cumulative distribution functions (cdf's) of some continuous random variables, say Z^{sup} and Z^{inf}, taking values in $(0,1)$, whose pdf's are given by $w_b^{sup}(z)$ and $w_b^{inf}(z)$, respectively.

For a parent distribution $G(x) = G(x; \boldsymbol{\xi})$, where $\boldsymbol{\xi}$ is a parameter vector, we propose the following generalized distributions:

$$G^{sup}(x;\boldsymbol{\xi},b) = \int_{-\infty}^{G(x)} w_b^{sup}(z)\,dz = W_b^{sup}[G(x)] = G(x) + G^b(x) - G^{b+1}(x) \quad (2.1)$$

and

$$G^{inf}(x;\boldsymbol{\xi},b) = \int_{-\infty}^{G(x)} w_b^{inf}(z)\,dz = W_b^{inf}[G(x)] = G(x)[1 - \overline{G}^b(x)], \quad (2.2)$$

where $\overline{G}(x) = 1 - G(x)$ is the survival function of the parent distribution. Observe that, if $b \in \mathbb{N}$, then the functions $G^b(x)$ and $1 - \overline{G}^b(x)$ represent the maximum and minimum distributions of "b" i.i.d. random variables with commom distribution $G(x)$. These distributions are also known as type I and type II Lehmann alternatives, respectively.

If $G(x)$ is absolutely continuous with pdf $g(x) = g(x;\boldsymbol{\xi})$ and support $\mathcal{X} = \{x \in \mathbb{R} : g(x) > 0\}$, then $G^{sup}(x)$ and $G^{inf}(x)$ are also absolutely continuous with support $\mathcal{X}$ and pdf's given by

$$g^{sup}(x;\boldsymbol{\xi},b) = g(x)\,w_b^{sup}[G(x)] = g(x) + h_b(x) - h_{b+1}(x) \quad (2.3)$$

and

$$g^{inf}(x;\boldsymbol{\xi},b) = g(x)\,w_b^{inf}[G(x)] = g(x)[1 - \overline{G}^b(x)] + G(x)\,\bar{h}_b(x), \quad (2.4)$$

where $h_b(x) = b\,g(x)\,G^{b-1}(x)$ and $\bar{h}_b(x) = b\,g(x)\,\overline{G}^{b-1}(x)$ are the pdf's corresponding to the $G^b(x)$ and $1 - \overline{G}^b(x)$ distributions, respectively. In this case, $g^{sup}(x)$ can be expressed as sums of exp-G densities.

Hereafter, a random variable X with pdf given by (2.3) is denoted by $X \sim G^{sup}(\boldsymbol{\xi},b)$. Equivalently, a random variable X with pdf given by (2.4) is denoted by $X \sim G^{inf}(\boldsymbol{\xi},b)$.

As $b \to \infty$, we have, for any fixed value of x,

$$\lim_{b\to\infty} G^b(x) = \begin{cases} 1, & \text{if } G(x) = 1, \\ 0, & \text{if } G(x) < 1, \end{cases} \qquad \lim_{b\to\infty} [1 - \overline{G}^b(x)] = \begin{cases} 1, & \text{if } G(x) > 0, \\ 0, & \text{if } G(x) = 0. \end{cases}$$

Therefore, from (2.1), (2.2), (2.3) and (2.4) we note that, for both families, $G(x)$ is the limiting distribution when $b \to \infty$, that is,

$$G(x) = \lim_{b\to\infty} G^{sup}(x) = \lim_{b\to\infty} G^{inf}(x) \text{ and } g(x) = \lim_{b\to\infty} g^{sup}(x) = \lim_{b\to\infty} g^{inf}(x). \quad (2.5)$$

For reasons that will be clear later, we call the families defined by (2.1) and (2.2) as supremum and infimum families of distributions, respectively.

2.2.1 Hazard rate and reverse hazard rate functions

In lifetime analysis, two very useful functions are the hrf $r(x) = g(x)/\overline{G}(x)$ and the rhrf $s(x) = g(x)/G(x)$. Consider the functions $\tau_b^{sup}(z) = w_b^{sup}(z)/[1 - W_b^{sup}(z)]$ and $\kappa_b^{sup}(z) = w_b^{sup}(z)/W_b^{sup}(z)$. For the supremum family of distributions, the hrf and rhrf are given by (MARSHALL; OLKIN, 1997, p. 32)

$$r^{sup}(x) = g(x)\,\tau_b^{sup}[G(x)] = \frac{g(x)}{\overline{G}(x)} + \frac{h_b(x)}{1 - G^b(x)} = r_Y(x) + r_Z(x) \qquad (2.6)$$

and

$$s^{sup}(x) = g(x)\,\kappa_b^{sup}[G(x)] = \frac{g(x) + h_b(x) - h_{b+1}(x)}{G(x) + G^b(x) - G^{b+1}(x)},$$

where $r_Y(x)$ and $r_Z(x)$ are the hrf's of $Y \sim G$ and $Z \sim G^b$, respectively.

Similarly, let $\tau_b^{inf}(z) = w_b^{inf}(z)/[1 - W_b^{inf}(z)]$ and $\kappa_b^{inf}(z) = w_b^{inf}(z)/W_b^{inf}(z)$. Then, the hrf and rhrf of the infimum family of distributions are given by

$$r^{inf}(x) = g(x)\,\tau_b^{inf}[G(x)] = \frac{g(x)[1 - \overline{G}^b(x)] + G(x)\,\bar{h}_b(x)}{1 - G(x)[1 - \overline{G}^b(x)]} \qquad (2.7)$$

and

$$s^{inf}(x) = g(x)\,\kappa_b^{inf}[G(x)] = \frac{g(x)}{G(x)} + \frac{\bar{h}_b(x)}{1 - \overline{G}^b(x)} = s_Y(x) + s_Z(x), \qquad (2.8)$$

where $s_Y(x)$ and $s_Z(x)$ are the rhrf's of $Y \sim G$ and $Z \sim [1 - \overline{G}^b(x)]$, respectively.

2.3 Motivations

In this section, we give some properties satisfied by the supremum and infimum families which motivated its introduction. Some of these properties are important in practice, since models with such characteristics have a wide range of applicability.

2.3.1 Physical motivation

From a physical view point, supremum and infimum families have the following interpretations. Given a cdf $G(x)$, consider a series system composed by two independent components such that the system fails if any of the components fail, and suppose that Y and Z denotes their life lengths. Suppose that $Y \sim G$ and $Z \sim G^b$, for $b > 0$.

This happens, for example, if $b \in \mathbb{N}$ and the component with life length Z is composed by b independent parallel subcomponents such that the component fails only if all the subcomponents fail, where each subcomponent has life length $Z_j \sim G, j = 1,\ldots,b$ (in this case, $Z = \max(Z_1,\ldots,Z_b)$). Let $X = \min(Y,Z)$. Then, X represents the life length of the entire system. We have that

$$P(X > x) = P(Y > x,Z > x) = P(Y > x)\, P(Z > x).$$

Thus, the cdf of X is

$$F_X(x) = 1 - \overline{G}(x)[1 - G^b(x)] = W_b^{sup}[G(x)],$$

which is given in (2.1).

Similarly, suposse a parallel system composed by two independent components and let Y and Z be their life lengths. Suppose that $Y \sim G$ and $Z \sim [1 - \overline{G}^b(x)]$, for $b > 0$. This can happen if $b \in \mathbb{N}$ and the component with life length Z is composed by b independent series subcomponents, where each subcomponent has life length $Z_j \sim G, j = 1,\ldots,b$ (that is, $Z = \min(Z_1,\ldots,Z_b)$). Thus, $X = \max(Y,Z)$ represents the life length of the entire system. We can write

$$P(X \leq x) = P(Y \leq x,Z \leq x) = P(Y \leq x)\, P(Z \leq x).$$

Therefore, the cdf of X is given by

$$F_X(x) = G(x)[1 - \overline{G}^b(x)] = W_b^{inf}[G(x)],$$

which is given in (2.2).

In terms of order statistics, if $X_{(n)}$ and $X_{(1)}$ represent the maximum and minimum of a random sample of size n from the distribution $G(x) = G(x,\boldsymbol{\xi})$, then

$$1 - G_{(n)}(x;\boldsymbol{\xi}) = \frac{1 - G^{sup}(x;\boldsymbol{\xi},n)}{\overline{G}(x;\boldsymbol{\xi})} \text{ and } G_{(1)}(x;\boldsymbol{\xi}) = \frac{G^{inf}(x;\boldsymbol{\xi},n)}{G(x;\boldsymbol{\xi})},$$

where

$$G_{(n)}(x) = G^n(x) \text{ and } G_{(1)}(x) = 1 - [1 - G(x)]^n$$

are the cdf's of $X_{(n)}$ and $X_{(1)}$, respectively.

2.3.2 Simple expressions in terms of exp-G densities

From equation (2.3), it is clear that the pdf of the supremum family is a simple sum of exp-G densities. To obtain an expression for the density of the infimum family given in (2.4) in terms of exp-G densities, we consider the following power series expansion, which converges for $|z| \leq 1$ and $\rho > -1$,

$$(1-z)^{\rho} = \sum_{j=0}^{\infty} \frac{(-1)^j\,(\rho)_j}{j!}\, z^j, \tag{2.9}$$

where $(\rho)_j = \rho\,(\rho-1)\,\cdots(\rho-j+1)$ is the falling factorial. Naturally, if $\rho \in \mathbb{N}$, the sum (2.9) ends in ρ and $(\rho)_j/j! = \binom{\rho}{j}$.

Based on (2.9), we have

$$G^{inf}(x) = G(x)[1-\overline{G}^{\,b}(x)] = G(x) - \sum_{j=0}^{\infty} \frac{(-1)^j\,(b)_j}{j!}\, G^{j+1}(x).$$

So, differentiating term to term, we obtain

$$g^{inf}(x) = g(x) - \sum_{j=0}^{\infty} \frac{(-1)^j\,(b)_j}{j!}\, h_{j+1}(x). \tag{2.10}$$

Therefore, the density $g^{inf}(x)$ can be expressed as a linear combination of exp-G densities.

2.3.3 Simple expressions for moments

Moments are fundamental in any statistical analysis. However, moments of generalized models not always have simple expressions. A motivation to introducing the supremum and infimum families is that they have simple expressions for the moments as sums of moments of exp-G distributions.

For $r \in \mathbb{N}$, the r-th ordinary moment of a random variable $X \sim G^{sup}(\boldsymbol{\xi},b)$ is given by

$$\mu_r^{sup} = \mathbb{E}(X^r) = \int_{-\infty}^{\infty} x^r\, g^{sup}(x)\, dx = \int_{-\infty}^{\infty} x^r\, g(x)\, w_b^{sup}[G(x)]\, dx.$$

From (2.3), we have

$$\mu_r^{sup} = \int_{-\infty}^{\infty} x^r\, g(x)\, dx + \int_{-\infty}^{\infty} x^r\, h_b(x)\, dx - \int_{-\infty}^{\infty} x^r\, h_{b+1}(x)\, dx,$$

and therefore

$$\mu_r^{sup} = \mathbb{E}(Z_1^r) + \mathbb{E}(Z_b^r) - \mathbb{E}(Z_{b+1}^r),$$

where $Z_\rho \sim G^\rho$, $\rho > 0$.

Similarly, the r-th ordinary moment of a random variable $X \sim G^{inf}(\boldsymbol{\xi}, b)$ is given by

$$\mu_r^{inf} = \mathbb{E}(X^r) = \int_{-\infty}^{\infty} x^r g^{inf}(x)\,dx = \int_{-\infty}^{\infty} x^r g(x)\, w_b^{inf}[G(x)]\,dx$$

and therefore from (2.4) gives

$$\mu_r^{inf} = \int_{-\infty}^{\infty} x^r g(x)[1 - \overline{G}^b(x)]\,dx + \int_{-\infty}^{\infty} x^r G(x)\,\bar{h}_b(x)\,dx.$$

If $\mathbb{E}(Z_1^r) < \infty$, then

$$\mathbb{E}(Z_j^r) = \int_{-\infty}^{\infty} x^r h_j(x)\,dx \le j \int_{-\infty}^{\infty} x^r g(x)\,dx = j\,\mathbb{E}(Z_1^r) < \infty, \quad \forall j \ge 1.$$

Thus, based on the linear combination (2.10) and assuming that $\mathbb{E}(Z_1^r) < \infty$, we obtain, by an application of the dominated convergence theorem,

$$\mu_r^{inf} = \mathbb{E}(Z_1^r) - \sum_{j=0}^{\infty} \frac{(-1)^j\,(b)_j}{(j+1)!}\,\mathbb{E}(Z_{j+1}^r) + b \sum_{j=0}^{\infty} \frac{(-1)^j (j+1)\,(b-1)_j}{(j+2)!}\,\mathbb{E}(Z_{j+2}^r).$$

2.3.4 Inducing bathtub shape in the hrf

In lifetime analysis, some shape properties of the hrf have important implications in practice. Many basic distributions used in lifetime analysis have a limited range of behaviour and can not represent all the situations in real applications. Thus, generalized distributions play a fundamental role in providing more flexible models that present hrf's with the classical shapes: increasing, decreasing, unimodal and bathtub. Among them, bathtub hazard rates have considerable intuitive appeal. For example, they may be useful for modelling lifetime rate of biological organisms (human life, for example), devices which come from a mixture of subcomponents of varying inherent strength, or series systems (LAI, 2013; MARSHALL; OLKIN, 1997).

Next, we will verify that the supremum family can induce bathtub shape in its hrf. It happens because this family can be expressed as sum (mixture) of exp-G distributions, or, from a physical approach, can be modeled as a series system. In fact, from (2.6), we have

$$r^{sup}(x) = r_Y(x) + r_Z(x),$$

where $Y \sim G$ and $Z \sim G^b$ are independent random variables. Thus, if $r_Y(x)$ is decreasing and $r_Z(x)$ is increasing, or vice versa, $r^{sup}(x)$ can be bathub shaped. Furthermore, if $r_Y(x)$ and $r_Z(x)$ are both convex, then $r^{sup}(x)$ must be convex, and therefore monotone or bathtub shaped. A more detailed discussion on bathtub hazard rates and its properties is given in Marshall & Olkin (1997), Section 4-D.

2.4 Some special G^{sup} and G^{inf} distributions

Next, we introduce some new models obtained by applying the methods discussed in Section 2.2 to parent distributions widely used in lifetime analysis. These parent distributions have in common that their hrf's do not have bathtub shape. We will see that the supremum family induces, in some cases, shape bathtub, besides both methods provide greater flexibility.

2.4.1 Supremum and infimum uniform distributions

Let $G(x) = x$ be the cdf of the uniform ($\mathcal{U}$) distribution in $(0,1)$ with pdf $g(x) = 1$. From (2.1), (2.3) and (2.6), a random variable $X \sim \mathcal{U}^{sup}(b)$ has cdf, pdf and hrf given by

$$G^{sup}(x;b) = W_b^{sup}(x) = x + (1-x)x^b, \tag{2.11}$$

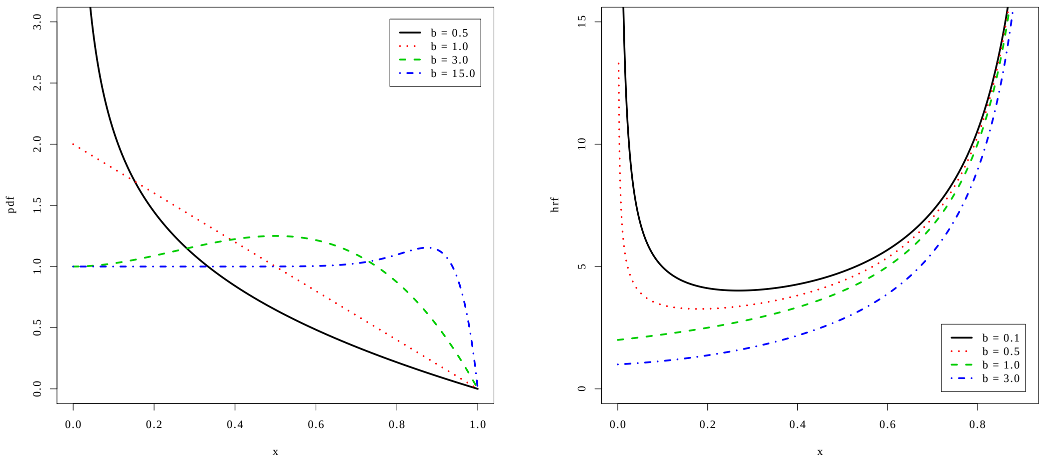
$$g^{sup}(x;b) = w_b^{sup}(x) = 1 - x^b + b\,(1-x)x^{b-1} \tag{2.12}$$

and

$$r^{sup}(x;b) = \tau_b^{sup}(x) = \frac{1}{1-x} + \frac{b\,x^{b-1}}{1-x^b},$$

for $0 < x < 1$ and $b > 0$. Also, from (2.2), (2.4) and (2.7), a random variable $X \sim \mathcal{U}^{inf}(b)$ has cdf, pdf and hrf given by

$$G^{inf}(x;b) = W_b^{inf}(x) = x\,[1 - (1-x)^b], \tag{2.13}$$



(a) pdf (b) hrf

Figure 2.1: Plots of the pdf and hrf of the $\mathcal{U}^{sup}(b)$ distribution for several values of b.

$$g^{inf}(x;b) = w_b^{inf}(x) = 1 - (1-x)^b + b\,x\,(1-x)^{b-1} \tag{2.14}$$

and

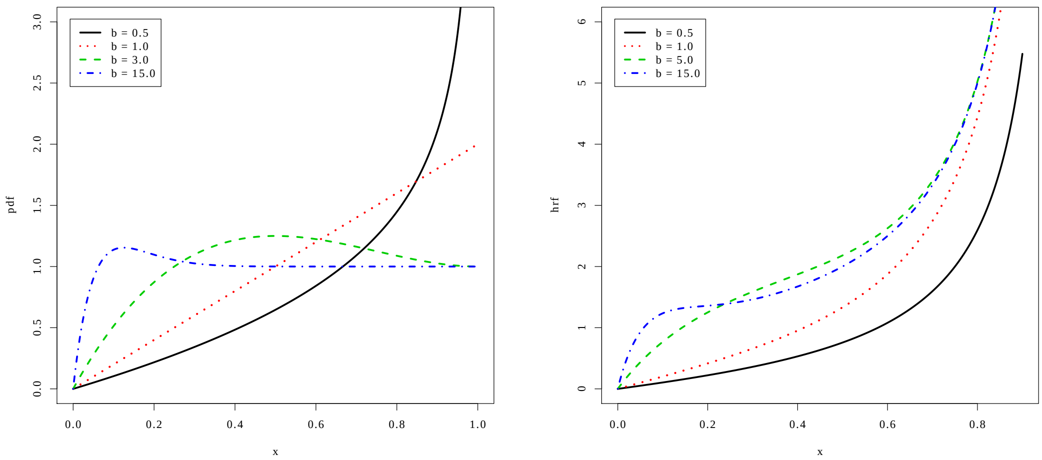
$$r^{inf}(x) = \tau_b^{inf}(x) = \frac{1 - (1-x)^b + b\,x\,(1-x)^{b-1}}{1 - x\,[1 - (1-x)^b]}.$$

In Figures 2.1 and 2.2, we display the pdf and hrf of the $\mathcal{U}^{sup}$ and $\mathcal{U}^{inf}$ distributions, respectively, for several values of b. The pdf can be decreasing or increasing when $0 < b \leq 1$ or unimodal when $b > 1$. The hrf can be increasing or have the classical bathtub shape and therefore this distribution can be appropriate for different applications in lifetime analysis.

2.4.2 Supremum and infimum Fréchet distributions

Consider the cdf and pdf of the Fréchet (Fr) distribution, which are given by $G(x;\alpha,\beta) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$ and $g(x;\alpha,\beta) = \frac{\beta}{x}\left(\frac{\alpha}{x}\right)^{\beta}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]$, respectively, for $x > 0$ and $\alpha,\beta > 0$. From (2.1), (2.3) and (2.6), a random variable $X \sim \mathrm{Fr}^{sup}(\alpha,\beta,b)$, $b > 0$, has cdf, pdf and hrf given, respectively, by

$$G^{sup}(x;\alpha,\beta,b) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] + \exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}, \tag{2.15}$$



(a) pdf (b) hrf

Figure 2.2: Plots of the pdf and hrf of the $\mathcal{U}^{inf}(b)$ distribution for several values of b.

$$g^{sup}(x;\alpha,\beta,b) = \frac{\beta}{x}\left(\frac{\alpha}{x}\right)^{\beta}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right] \times \left\{1-\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right] - b\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}\right\} \tag{2.16}$$

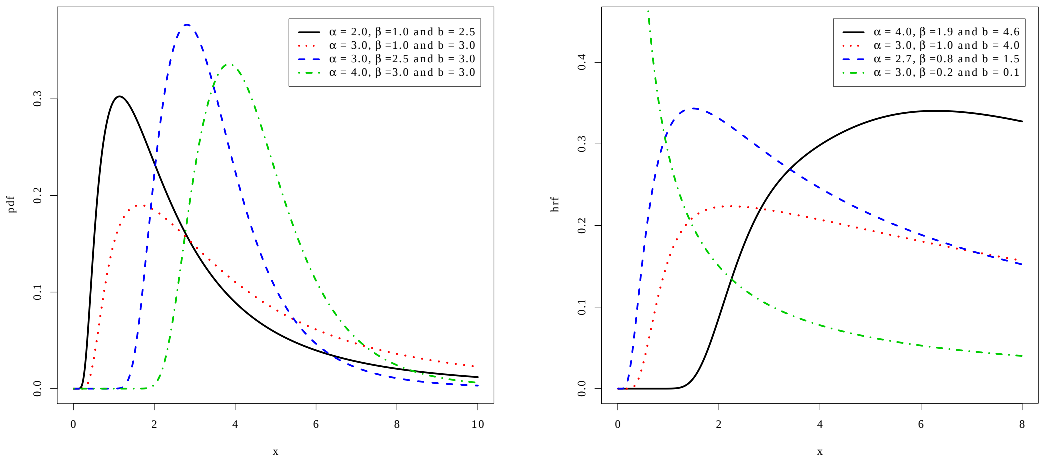
and

$$r^{sup}(x;\alpha,\beta,b) = \frac{\frac{\beta}{x}\left(\frac{\alpha}{x}\right)^{\beta}\left\{1-\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right] - b\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}\right\}}{\left\{1-\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}\left\{1-\exp\left[-b\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}}.$$

Equivalently, from (2.2), (2.4) and (2.7), a random variable $X \sim \text{Fr}^{inf}(\alpha,\beta,b)$, $b>0$, has cdf, pdf and hrf, respectively, given by

$$G^{inf}(\alpha,\beta,b) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b}\right\}, \tag{2.17}$$

$$g^{inf}(\alpha,\beta,b) = \frac{\beta}{x}\left(\frac{\alpha}{x}\right)^{\beta}\exp\left[-2\left(\frac{\alpha}{x}\right)^{\beta}\right] \times \left\{b\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1} + \exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b}\right\}\right\} \tag{2.18}$$



(a) pdf (b) hrf

Figure 2.3: Plots of the pdf and hrf of the $\mathrm{Fr}^{sup}(\alpha,\beta,b)$ distribution for selected parameters.

and

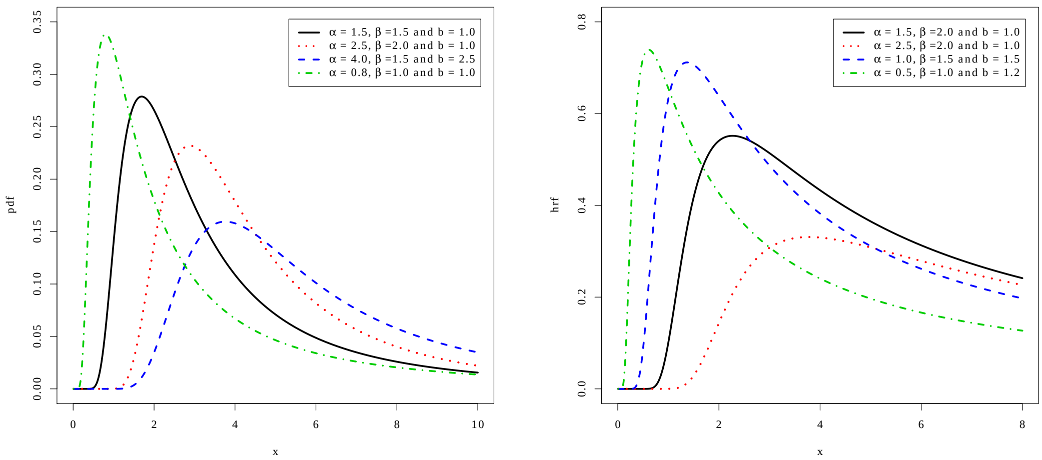
$$r^{inf}(\alpha,\beta,b)=\frac{\frac{\beta}{x}\left(\frac{\alpha}{x}\right)^{\beta}\left\{b\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b-1}+\exp\left[\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b}\right\}\right\}}{\exp\left[2\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}\left\{1-\left\{1-\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{b}\right\}}.$$

In Figures 2.3 and 2.4, we display the pdf and hrf of the supremum and infimum Fréchet distributions for selected parameters, respectively. They show that the pdf is unimodal and the hrf can have decreasing or unimodal shapes. Therefore, these distributions can be appropriate for different applications in lifetime analysis (see Section 2.10).

2.4.3 Supremum and infimum Weibull distributions

Let $G(x;\alpha,\beta)=1-\mathrm{e}^{-\alpha x^{\beta}}$ be the cdf of the Weibull (W) distribution with support $x>0$ and parameters $\alpha,\beta>0$ and let $g(x;\alpha,\beta)=\alpha\,\beta\,x^{\beta-1}\,\mathrm{e}^{-\alpha x^{\beta}}$ be the corresponding pdf. From (2.1), (2.3) and (2.6), a random variable $X\sim\mathrm{W}^{sup}(\alpha,\beta,b)$, $\alpha,\beta,b>0$, has cdf, pdf and hrf, respectively, given by

$$G^{sup}(x;\alpha,\beta,b)=1-\mathrm{e}^{-\alpha\,x^{\beta}}+\mathrm{e}^{-\alpha\,x^{\beta}}\left(1-\mathrm{e}^{-\alpha\,x^{\beta}}\right)^{b}, \qquad (2.19)$$



(a) pdf (b) hrf

Figure 2.4: Plots of the pdf and hrf of the $\mathrm{Fr}^{inf}(\alpha,\beta,b)$ distribution for selected parameters.

$$g^{sup}(x;\alpha,\beta,b) = \alpha\,\beta\,x^{\beta-1}\,\mathrm{e}^{-\alpha x^{\beta}}\left[1-\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{b}+b\,\mathrm{e}^{-\alpha x^{\beta}}\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{b-1}\right] \tag{2.20}$$

and

$$r^{sup}(x;\alpha,\beta,b) = \alpha\,\beta\,x^{\beta-1} + \frac{\alpha\,\beta\,b\,x^{\beta-1}\,\mathrm{e}^{-\alpha x^{\beta}}\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{b-1}}{1-\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)^{b}}.$$

Equivalently, from (2.2), (2.4) and (2.7), the infimum Weibull distribution has cdf, pdf and hrf, respectively, given by

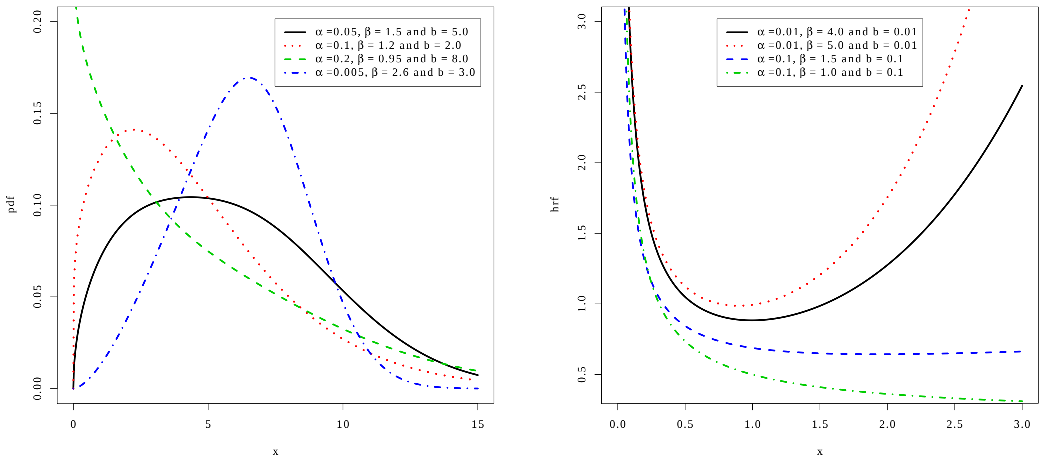
$$G^{inf}(x;\alpha,\beta,b) = \left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)\left(1-\mathrm{e}^{-\alpha b x^{\beta}}\right), \tag{2.21}$$

$$g^{inf}(x;\alpha,\beta,b) = \alpha\,\beta\,x^{\beta-1}\,\mathrm{e}^{-\alpha x^{\beta}}\left[1+b\,\mathrm{e}^{-\alpha(b-1)x^{\beta}}-(b+1)\,\mathrm{e}^{-\alpha b x^{\beta}}\right] \tag{2.22}$$

and

$$r^{inf}(x;\alpha,\beta,b) = \frac{\alpha\,\beta\,x^{\beta-1}\,\mathrm{e}^{-\alpha x^{\beta}}\left[1+b\,\mathrm{e}^{-\alpha(b-1)x^{\beta}}-(b+1)\,\mathrm{e}^{-\alpha b x^{\beta}}\right]}{1-\left(1-\mathrm{e}^{-\alpha x^{\beta}}\right)\left(1-\mathrm{e}^{-\alpha b x^{\beta}}\right)}.$$

Figures 2.5 and 2.6 display the pdf and hrf of the supremum and infimum Weibull



(a) pdf (b) hrf

Figure 2.5: Plots of the pdf and hrf of the W$^{sup}(\alpha,\beta,b)$ distribution for selected parameters.

distributions for selected parameters, respectively. We note that the pdf can be decreasing or unimodal and that the hrf can have the classical shapes: increasing, decreasing and bathtub. Therefore, these distributions can be appropriate for different applications in lifetime analysis.

2.4.4 Supremum and infimum Lomax distributions

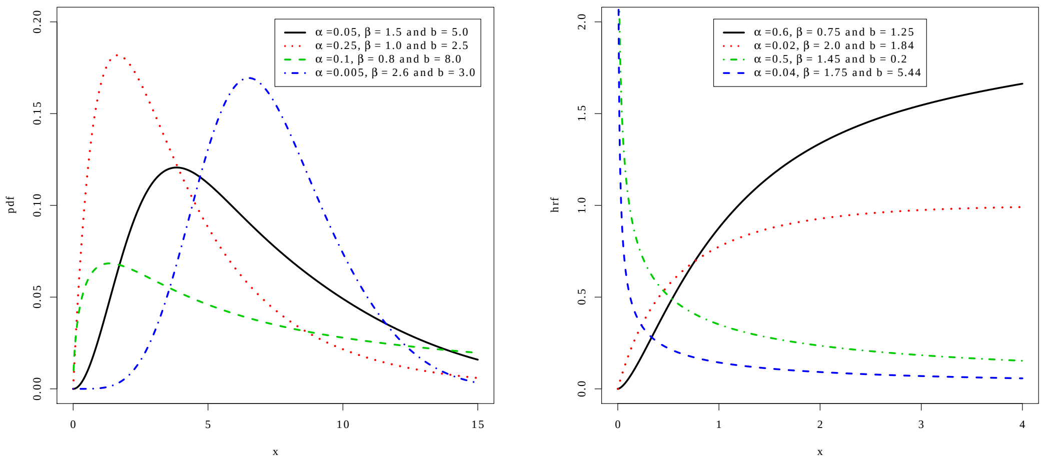
If $G(x;\alpha,\beta) = 1-(1+\beta x)^{-\alpha}$ is the Lomax (Lo) cdf with support $x>0$, parameters $\alpha,\beta>0$ and corresponding pdf $g(x;\alpha,\beta) = \alpha\beta(1+\beta x)^{-(\alpha+1)}$, then the cdf, pdf and hrf of the supremum and infimum Lomax distributions are given, respectively, by

$$G^{sup}(x;\alpha,\beta,b) = (1+\beta x)^{-\alpha}\{-1+(1+\beta x)^{\alpha}+[1-(1+\beta x)^{-\alpha}]^{b}\},$$

$$g^{sup}(x;\alpha,\beta,b) = -\frac{\alpha\,\beta\,(1+\beta x)^{-(\alpha+1)}\{[1-(1+\beta x)^{\alpha}]\{1-[1-(1+\beta x)^{-\alpha}]^{b}\}-b[1-(1+\beta x)^{-\alpha}]^{b}\}}{(1+\beta x)^{\alpha}-1},$$

$$r^{sup}(x;\alpha,\beta,b) = \frac{\alpha\,\beta}{1+\beta x}+\frac{\alpha\,\beta\,b\,(1+\beta x)^{-(\alpha+1)}\,[1-(1+\beta x)^{-\alpha}]^{b-1}}{1-[1-(1+\beta x)^{-\alpha}]^{b}},$$

$$G^{inf}(x;\alpha,\beta,b) = \left[1-(1+\beta x)^{-\alpha}\right]\left[1-(1+\beta x)^{-b\alpha}\right],$$



(a) pdf (b) hrf

Figure 2.6: Plots of the pdf and hrf of the W$^{inf}(\alpha,\beta,b)$ distribution for selected parameters.

$$g^{inf}(x;\alpha,\beta,b) = \alpha\,\beta\,(1+\beta x)^{-(\alpha+1)}\left[1+b\,(1+\beta x)^{-(b-1)\alpha}-(b+1)\,(1+\beta x)^{-b\alpha}\right]$$

and

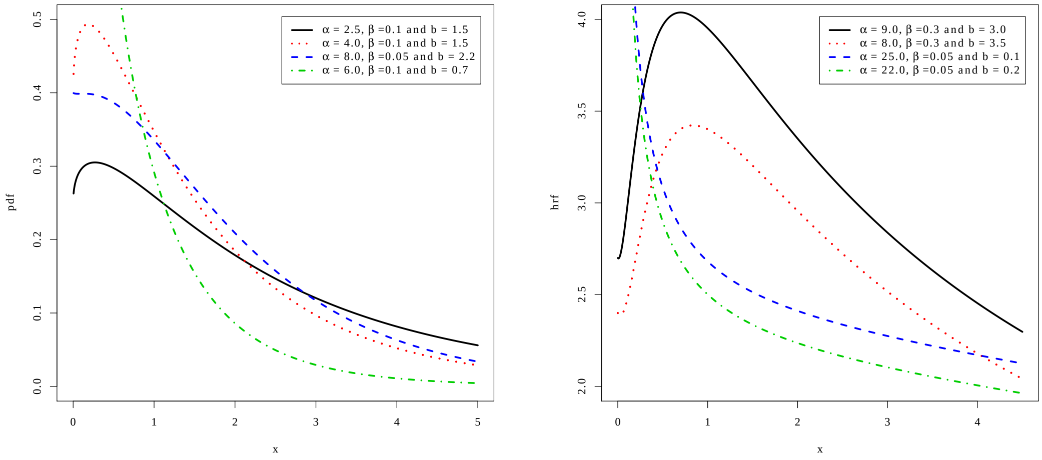
$$r^{inf}(x;\alpha,\beta,b) = \frac{\alpha\,\beta\,(1+\beta x)^{-(\alpha+1)}\left[1+b\,(1+\beta x)^{-(b-1)\alpha}-(b+1)\,(1+\beta x)^{-b\alpha}\right]}{1-\left[1-(1+\beta x)^{-\alpha}\right]\left[1-(1+\beta x)^{-b\alpha}\right]}.$$

Figures 2.7 and 2.8 display the pdf and hrf of the supremum and infimum Lomax distributions for selected parameters, respectively. The hrf can have the classical shapes as decreasing and unimodal. Therefore, these distributions can be appropriate for different applications in lifetime analysis.

2.4.5 Supremum and infimum log-logistic distributions

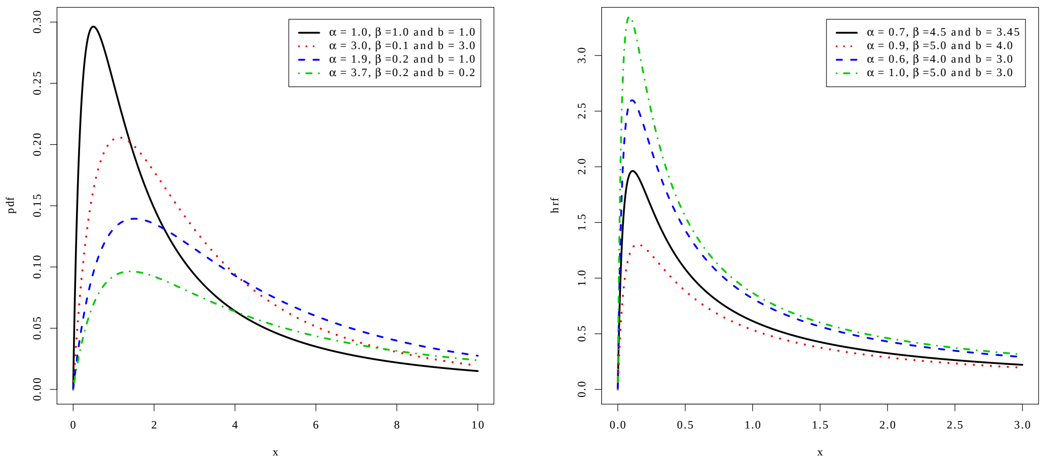
Consider the log-logistic (LL) distribution with support $x > 0$ and parameters $\alpha,\beta > 0$, which has cdf $G(x;\alpha,\beta) = x^\beta/(\alpha^\beta + x^\beta)$ and pdf $g(x;\alpha,\beta) = (\beta/\alpha)(x/\alpha)^{\beta-1}/[1 + (x/\alpha)^\beta]^2$. So, the cdf, pdf and hrf of the supremum and infimum log-logistic distributions are given, respectively, by

$$G^{sup}(x;\alpha,\beta,b) = \frac{x^\beta + \alpha^\beta\left(\frac{x^\beta}{\alpha^\beta + x^\beta}\right)^b}{\alpha^\beta + x^\beta},$$



(a) pdf (b) hrf

Figure 2.7: Plots of the pdf and hrf of the $\mathrm{Lo}^{sup}(\alpha,\beta,b)$ distribution for selected parameters.



(a) pdf (b) hrf

Figure 2.8: Plots of the pdf and hrf of the $\mathrm{Lo}^{inf}(\alpha,\beta,b)$ distribution for selected parameters.

$$g^{sup}(x;\alpha,\beta,b) = \frac{\alpha^{\beta}\beta\left\{x^{\beta}\left[1-\left(\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right]+\alpha^{\beta}b\left(\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right\}}{x(\alpha^{\beta}+x^{\beta})^{2}},$$

$$r^{sup}(x;\alpha,\beta,b) = \frac{\beta\left\{x^{\beta}\left[1-\left(\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right]+\alpha^{\beta}b\left(\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right\}}{x(\alpha^{\beta}+x^{\beta})\left[1-\left(\frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right]},$$

$$G^{inf}(x;\alpha,\beta,b) = \frac{x^{\beta}}{\alpha^{\beta}+x^{\beta}}\left[1-\left(\frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right],$$

$$g^{inf}(x;\alpha,\beta,b) = \frac{\beta\,x^{\beta-1}}{(\alpha^{\beta}+x^{\beta})^{2}}\left\{b\,x^{\beta}\left(\frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}+\alpha^{\beta}\left[1-\left(\frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right]\right\}$$

and

$$r^{inf}(x;\alpha,\beta,b) = \frac{\beta\,x^{\beta-1}\left\{b\,x^{\beta}\left(\frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}+\alpha^{\beta}\left[1-\left(\frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right]\right\}}{(\alpha^{\beta}+x^{\beta})\left[\alpha^{\beta}+x^{\beta}\left(\frac{\alpha^{\beta}}{\alpha^{\beta}+x^{\beta}}\right)^{b}\right]}.$$

Figures 2.9 and 2.10 display the pdf and hrf of the supremum and infimum log-logistic distributions for selected parameters. The classical shapes for the hrf as increasing, decreasing, bathtub and unimodal reveal that these distributions can be appropriate for different applications in lifetime analysis (see Section 2.10).

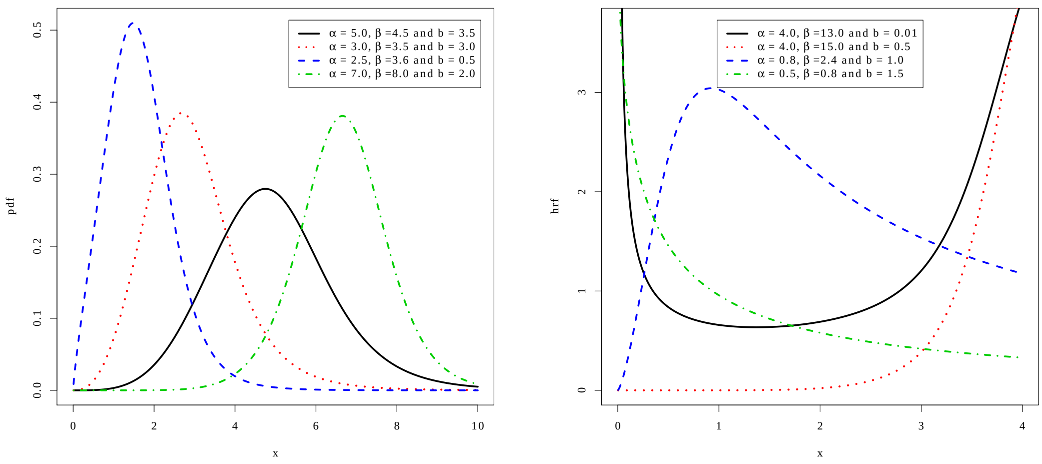
2.5 Shapes of the density and hazard rate functions

2.5.1 Pdf and hrf of the G^{sup} family

For the absolutely continuous case, the shape of the pdf (2.3) can be described analytically by examining the roots of the equation

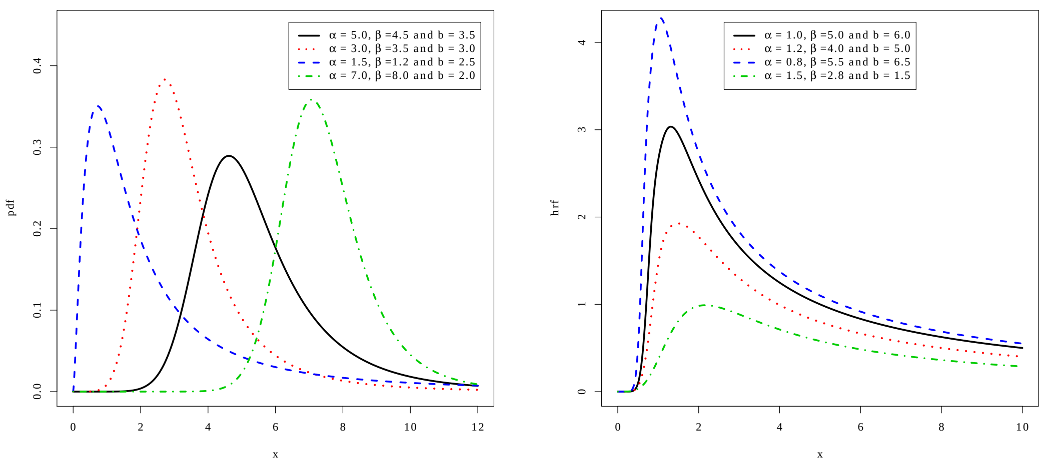
$$\frac{d}{dx}\,g^{sup}(x) = g'(x)\,w_b^{sup}[G(x)] + g^2(x)\,\frac{d}{dx}\,w_b^{sup}[G(x)]$$

and analyzing its limits when $x \to 0$ or $x \to \infty$. Cleary, since $g(x) \geq 0$ is continuous and integrable and $\lim_{x\to\infty} w_b^{sup}[G(x)] = 0$, then $\lim_{x\to\infty} g^{sup}(x) = 0$.



(a) pdf (b) hrf

Figure 2.9: Plots of the pdf and hrf of the $\mathrm{LL}^{sup}(\alpha,\beta,b)$ distribution for selected parameters.



(a) pdf (b) hrf

Figure 2.10: Plots of the pdf and hrf of the $\mathrm{LL}^{inf}(\alpha,\beta,b)$ distribution for selected parameters.

Alternativelly, the critical points of $g^{sup}(x)$ can be obtained by examining the roots of the equation

$$\frac{d}{dx}\log[g^{sup}(x)] = \frac{d}{dx}\log[g(x)] + \frac{d}{dx}\log\{w_b^{sup}[G(x)]\}. \tag{2.23}$$

If $x = x_0$ is a root in (2.23), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\psi(x_0) < 0$, $\psi(x_0) > 0$ or $\psi(x_0) = 0$, where

$$\psi(x) = \frac{d^2}{dx^2}\log[g^{sup}(x)] = \frac{d^2}{dx^2}\log[g(x)] + \frac{d^2}{dx^2}\log\{w_b^{sup}[G(x)]\}.$$

From (2.6), the shape of the hrf can be described analytically by examining the roots of the equation

$$\frac{d}{dx}r^{sup}(x) = r'_Y(x) + r'_Z(x).$$

Alternativelly, the critical points of $r^{sup}(x)$ can be obtained by examining the roots of the equation

$$\begin{aligned}\frac{d}{dx}\log[r^{sup}(x)] &= \frac{d}{dx}\log[g^{sup}(x)] + r^{sup}(x)\\ &= \frac{d}{dx}\log[g(x)] + \frac{d}{dx}\log\{w_b^{sup}[G(x)]\} + r^{sup}(x).\end{aligned} \tag{2.24}$$

If $x = x_0$ is a root in (2.24), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\zeta(x_0) < 0$, $\zeta(x_0) > 0$ or $\zeta(x_0) = 0$, where

$$\zeta(x) = \frac{d^2}{dx^2}\log[r^{sup}(x)] = \frac{d^2}{dx^2}\log[g(x)] + \frac{d^2}{dx^2}\log\{w_b^{sup}[G(x)]\} + \frac{d}{dx}r^{sup}(x).$$

2.5.2 Pdf and rhrf of the G^{inf} family

For the absolutely continuous case, the shape of the pdf (2.4) can be described analytically by examining the roots of the equation

$$\frac{d}{dx}g^{inf}(x) = g'(x)\,w_b^{inf}[G(x)] + g^2(x)\,\frac{d}{dx}\,w_b^{inf}[G(x)]$$

and analyzing its limits when $x \to 0$ or $x \to \infty$. Since $g(x) \geq 0$ is continuous and integrable and $\lim_{x\to\infty} w_b^{inf}[G(x)] = 1$, then $\lim_{x\to\infty} g^{inf}(x) = 0$.

Alternativelly, the critical points of $g^{inf}(x)$ can be obtained by examining the roots of the equation

$$\frac{d}{dx}\log[g^{inf}(x)] = \frac{d}{dx}\log[g(x)] + \frac{d}{dx}\log\{w_b^{inf}[G(x)]\}. \tag{2.25}$$

If $x = x_0$ is a root in (2.25), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\psi(x_0) < 0$, $\psi(x_0) > 0$ or $\psi(x_0) = 0$, where

$$\psi(x) = \frac{d^2}{dx^2}\log[g^{inf}(x)] = \frac{d^2}{dx^2}\log[g(x)] + \frac{d^2}{dx^2}\log\{w_b^{inf}[G(x)]\}.$$

From (2.8), the shape of the rhrf can be described analytically by examining the roots of the equation

$$\frac{d}{dx}s^{inf}(x) = s'_Y(x) + s'_Z(x).$$

Alternativelly, the critical points of $s^{inf}(x)$ can be obtained by examining the roots of the equation

$$\begin{aligned}\frac{d}{dx}\log[s^{inf}(x)] &= \frac{d}{dx}\log[g^{inf}(x)] - s^{inf}(x)\\ &= \frac{d}{dx}\log[g(x)] + \frac{d}{dx}\log\{w_b^{inf}[G(x)]\} - s^{inf}(x).\end{aligned} \tag{2.26}$$

If $x = x_0$ is a root in (2.26), then it corresponds to a local maximum, a local minimum or a point of inflexion depending on whether $\zeta(x_0) < 0$, $\zeta(x_0) > 0$ or $\zeta(x_0) = 0$, where

$$\zeta(x) = \frac{d^2}{dx^2}\log[s^{inf}(x)] = \frac{d^2}{dx^2}\log[g(x)] + \frac{d^2}{dx^2}\log\{w_b^{inf}[G(x)]\} - \frac{d}{dx}s^{inf}(x).$$

2.6 Properties

Next, we obtain some properties of the supremum and infimum families and provide a formal proof of the equalities (2.5).

PROPOSITION 1 *Let $\mathcal{X} = \{x \in \mathbb{R} : g(x;\boldsymbol{\xi}) > 0\}$ be the support of the parent distribution, with parameter space Θ. Then,*

a) $G^{inf}(x;\boldsymbol{\xi},b) \leq G(x;\boldsymbol{\xi}) \leq G^{sup}(x;\boldsymbol{\xi},b)$*, for all* $x \in \mathcal{X}$, $\boldsymbol{\xi} \in \Theta$, $b > 0$.

b) The supremum and infimum families are decreasing and increasing monotonic *families with respect to b, respectively. That is, for all $j > 0$,*

$$G^{sup}(x;\boldsymbol{\xi},b+j) \leq G^{sup}(x;\boldsymbol{\xi},b), \quad G^{inf}(x;\boldsymbol{\xi},b) \leq G^{inf}(x;\boldsymbol{\xi},b+j),$$

for all $x \in \mathcal{X}$, $\boldsymbol{\xi} \in \Theta$ and $b > 0$.

c) $G^{sup}(x;\boldsymbol{\xi},b) \downarrow G(x;\boldsymbol{\xi})$ and $G^{inf}(x;\boldsymbol{\xi},b) \uparrow G(x;\boldsymbol{\xi})$ when $b \uparrow \infty$, for all $x \in \mathcal{X}$, $\boldsymbol{\xi} \in \Theta$.

Proof: From the equalities (2.1) and (2.2), we can write $G^{sup}(x) = G(x) + B_b^{sup}(x)$ and $G^{inf}(x) = G(x) - B_b^{inf}(x)$, where

$$B_b^{sup}(x) = G^b(x) - G^{b+1}(x) = \overline{G}(x)\, G^b(x), \quad B_b^{inf}(x) = G(x)\overline{G}^b(x).$$

It is clear that $B_b^{sup}(x) \geq 0$ and $B_b^{inf}(x) \geq 0$ for all $x \in \mathcal{X}$, $\boldsymbol{\xi} \in \Theta$ and $b > 0$, which implies *(a)*.

Next, for $x \in \mathcal{X}$ and $b > 0$ fixed, observe that $0 \leq G(x) \leq 1$ implies in $G^{b+j}(x) \leq G^b(x)$ and $\overline{G}^{b+j}(x) \leq \overline{G}^b(x)$ for all $j > 0$ and, therefore, gives $B_{b+j}^{sup}(x) \leq B_b^{sup}(x)$ and $B_{b+j}^{inf}(x) \leq B_b^{inf}(x)$. Thus, these conditions imply *(b)*.

Finally, observe that, if $G(x) = 0$ or $G(x) = 1$, then $B_b^{sup}(x) = 0$ and $B_b^{inf}(x) = 0$. On the other hand, if $0 < G(x) < 1$, then $B_b^{sup}(x) \downarrow 0$ and $B_b^{inf}(x) \downarrow 0$ when $b \uparrow \infty$. It proves *(c)*.

■

2.7 Generating random variates

In this section, we present a general method to obtain random variates from the families G^{sup} and G^{inf} for the absolutely continuous case. Suppose first that the parent distribution $G(\cdot)$ has quantile function (qf) $Q(\cdot)$. If u is an observation of the uniform distribution $\mathcal{U}(0,1)$, by the inversion method we have that $y = Q(u)$, $z = Q(u^{1/b})$ and $z' = Q(1-(1-u)^{1/b})$ are observations from the distributions G, G^b and $1 - \overline{G}^b(x)$, respectively. Therefore, from the results of Section 2.3.1, we can generate random variates $x_1, \ldots, x_n$ from the G^{sup} and G^{inf} families as detailed in the algorithms below.

If the parent distribution $G(\cdot)$ does not have qf in closed-form, the variates y_i, z_i and z_i' in the steps 2 and 3 can be obtained by solving the following equations iteratively:

$$\begin{aligned} u_i &= G(y_i), \quad i = 1, \ldots, n, \\ v_i^{1/b} &= G(z_i), \quad i = 1, \ldots, n, \\ 1-(1-v_i')^{1/b} &= G(z_i'), \quad i = 1, \ldots, n. \end{aligned}$$

`I)Generating random variates` $x_1,\ldots,x_n$ `from the` G^{sup} `family:`

1 For each $i = 1,\ldots,n$, get u_i and v_i independently from the uniform distribution $\mathcal{U}(0,1)$.

2 Calculate $y_i = Q(u_i)$.

3 Calculate $z_i = Q(v_i^{1/b})$.

4 Finally, calculate $x_i = \min(y_i,z_i)$.

`II)Generating random variates` $x_1,\ldots,x_n$ `from the` G^{inf} `family:`

1 For each $i = 1,\ldots,n$, get u_i and v_i' independently from the uniform distribution $\mathcal{U}(0,1)$.

2 Calculate $y_i = Q(u_i)$.

3 Calculate $z_i' = Q[1-(1-v_i')^{1/b}]$.

4 Finally, calculate $x_i = \max(y_i,z_i')$.

2.8 Maximum Likelihood Estimation

Several approaches for parameter estimation were proposed in the statistical literature but the maximum likelihood method is the most commonly employed. The MLE's enjoy desirable properties for constructing confidence intervals. In this section, we consider the estimation of the parameters of the G^{sup} and G^{inf} families by this method. Let $\boldsymbol{x} = (x_1,\ldots,x_n)^\top$ be a sample of size n from $X \sim G^{sup}(\boldsymbol{\xi},b)$ and let $\boldsymbol{\theta} = (\boldsymbol{\xi}^\top,b)^\top$ be the parameter vector. The log-likelihood for $\boldsymbol{\theta}$ based on the sample $\boldsymbol{x}$, denoted by $\ell_{g^{sup}}(\boldsymbol{\theta};\boldsymbol{x})$, is given by

$$\ell_{g^{sup}}(\boldsymbol{\theta};\boldsymbol{x}) = \ell_g(\boldsymbol{\xi};\boldsymbol{x}) + \sum_{i=1}^{n} \log\{w_b^{sup}[G(x_i,\boldsymbol{\xi})]\}, \tag{2.27}$$

where $\ell_g(\boldsymbol{\xi};\boldsymbol{x})$ is the log-likelihood corresponding to the parent distribution.

For $0 < G(x) < 1$, we have that $\lim_{b\to\infty} w_b^{sup}[G(x)] = 1$. Thus, it implies that $\lim_{b\to\infty} \ell_{g^{sup}}(\boldsymbol{\theta};\boldsymbol{x}) = \ell_g(\boldsymbol{\xi};\boldsymbol{x})$. This result coincides with the properties obtained in Proposition 1.

The MLE $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ can be obtained by maximizing (2.27) directly by using the `SAS` (`PROC NLMIXED`), `R` (`optim` and `MaxLik` functions) or the `Ox` program (sub-routine `MaxBFGS`). Details for fitting univariate distributions using maximum likelihood in `R` for censored or non censored data can be obtained at

`http://www.inside-r.org/packages/cran/fitdistrplus/docs/mledist.`

Alternatively, we can obtain the components of the score vector $\boldsymbol{U}_{\boldsymbol{\theta}}^{sup} = (U_{\boldsymbol{\xi}}^\top,U_b)^\top$

and set them to zero. They are given by

$$U_{\boldsymbol{\xi}} = \tilde{U}_{\boldsymbol{\xi}} + \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\xi}} \log\{w_b^{sup}[G(x_i,\boldsymbol{\xi})]\},$$
$$U_b = \sum_{i=1}^{n} \frac{\partial}{\partial b} \log\{w_b^{sup}[G(x_i,\boldsymbol{\xi})]\},$$

where $\tilde{U}_{\boldsymbol{\xi}} = \frac{\partial}{\partial \boldsymbol{\xi}} \ell_g(\boldsymbol{\xi}; \boldsymbol{x})$ is the score vector corresponding to the parent distribution.

Similarly, if $X \sim G^{inf}(\boldsymbol{\xi},b)$, the log-likelihood $\ell_{g^{inf}}(\boldsymbol{\theta}; \boldsymbol{x})$ is given by

$$\ell_{g^{inf}}(\boldsymbol{\theta}; \boldsymbol{x}) = \ell_g(\boldsymbol{\xi}; \boldsymbol{x}) + \sum_{i=1}^{n} \log\{w_b^{inf}[G(x_i,\boldsymbol{\xi})]\}.$$

Also, since $\lim_{b\to\infty} w_b^{inf}[G(x)] = 1$, we have that $\lim_{b\to\infty} \ell_{g^{inf}}(\boldsymbol{\theta}; \boldsymbol{x}) = \ell_g(\boldsymbol{\xi}; \boldsymbol{x})$.

The components of the score vector $\boldsymbol{U}_{\boldsymbol{\theta}}^{inf} = (U_{\boldsymbol{\xi}}^\top, U_b)^\top$ are given by

$$U_{\boldsymbol{\xi}} = \tilde{U}_{\boldsymbol{\xi}} + \sum_{i=1}^{n} \frac{\partial}{\partial \boldsymbol{\xi}} \log\{w_b^{inf}[G(x_i,\boldsymbol{\xi})]\},$$
$$U_b = \sum_{i=1}^{n} \frac{\partial}{\partial b} \log\{w_b^{inf}[G(x_i,\boldsymbol{\xi})]\}.$$

For all cases, the MLE $\hat{\boldsymbol{\theta}}_n$ can be determined by setting $U_{\boldsymbol{\xi}} = \mathbf{0}$ and $U_b = 0$ and by solving these equations simultaneously. If they can not be solved in closed-form, numerical iterative methods, such as Newton-Raphson type algorithms, can be applied.

2.8.1 Non-nested hypotheses tests

In many situations, it is of interest analyze how significant are the induced parameters of an extended or generalized distribution in modeling a data set. For example, we can testing the null hypotesis that the parent distribution gives the best fit to the data set versus the alternative that the generalized distribution is the best model. When the null hypothesis is nested within the alternative, standard classical procedures such as the likelihood ratio (LR) test can be used. However, this is not the case for the supremum and infimum families. In fact, the parent family $G_{\boldsymbol{\xi}} = \{g(x; \boldsymbol{\xi}) \,:\, \boldsymbol{\xi} \in \Theta\}$ and the families $G_{\boldsymbol{\theta}}^{sup} = \{g^{sup}(x; \boldsymbol{\theta}) \,:\, \boldsymbol{\theta} \in \Theta \times \mathbb{R}^+\}$ and $G_{\boldsymbol{\theta}}^{inf} = \{g^{inf}(x; \boldsymbol{\theta}) \,:\, \boldsymbol{\theta} \in \Theta \times \mathbb{R}^+\}$ are strictly non-nested in the sense that

$$G_{\boldsymbol{\theta}}^{sup} \cap G_{\boldsymbol{\xi}} = \varnothing \;\text{ and }\; G_{\boldsymbol{\theta}}^{inf} \cap G_{\boldsymbol{\xi}} = \varnothing.$$

Several authors have proposed in the literature procedures for testing non-nested hypotheses. Here, we consider the approach based on the generalized LR (GLR) test introduced by Vuong (1989).

Let $x_1,\ldots,x_n$ be a random sample from a distribution with density $f_0(x)$ unknown. Considering the G^{sup} family, define

$$D_g^{sup} = \mathbb{E}_{f_0}\{\log[g^{sup}(x;\boldsymbol{\theta}_*)] - \log[g(x;\boldsymbol{\xi}_*)]\},$$

where $\mathbb{E}_{f_0}(\cdot)$ denotes the expectation with respect to the true density $f_0(x)$ and

$$\boldsymbol{\theta}_* = \arg\max_{\boldsymbol{\theta}} \mathbb{E}_{f_0}\{\ell_{g^{sup}}(\boldsymbol{\theta})\}, \quad \boldsymbol{\xi}_* = \arg\max_{\boldsymbol{\xi}} \mathbb{E}_{f_0}\{\ell_g(\boldsymbol{\xi})\}$$

are known as the pseudo-true values of $\boldsymbol{\theta}$ and $\boldsymbol{\xi}$, respectively (PESARAN; ULLOA, 2008).

Based on the Kullback-Leibler information criterion (KULLBACK, 1997), a selection criterion can be defined to using the quantity D_g^{sup}. In fact, consider the following hypotheses and their definitions:

a) $\mathcal{H}_0: \ D_g^{sup} = 0,$ meaning that the models $G_{\boldsymbol{\theta}}^{sup}$ and $G_{\boldsymbol{\xi}}$ are equivalent, against

b) $\mathcal{H}_{g^{sup}}: \ D_g^{sup} > 0,$ meaning that the model $G_{\boldsymbol{\theta}}^{sup}$ is better than $G_{\boldsymbol{\xi}}$, or

c) $\mathcal{H}_g: \ D_g^{sup} < 0,$ meaning that the model $G_{\boldsymbol{\theta}}^{sup}$ is worse than $G_{\boldsymbol{\xi}}$.

To test these hypotheses, consider the GLR statistic, defined by

$$\mathrm{GLR}_n = \frac{1}{\sqrt{n}\,\hat{\boldsymbol{\omega}}_n}\,\mathrm{LR}_n(\hat{\boldsymbol{\theta}}_n,\hat{\boldsymbol{\xi}}_n),$$

where $\mathrm{LR}_n(\hat{\boldsymbol{\theta}}_n,\hat{\boldsymbol{\xi}}_n) = \ell_{g^{sup}}(\hat{\boldsymbol{\theta}}_n) - \ell_g(\hat{\boldsymbol{\xi}}_n)$, $\hat{\boldsymbol{\theta}}_n$ and $\hat{\boldsymbol{\xi}}_n$ are the MLE's of the models $G_{\boldsymbol{\theta}}^{sup}$ and $G_{\boldsymbol{\xi}}$, respectively, and

$$\hat{\boldsymbol{\omega}}_n^2 = \frac{1}{n}\sum_{i=1}^{n}\Big\{\log[g^{sup}(x_i;\hat{\boldsymbol{\theta}}_n)] - \log[g(x_i;\hat{\boldsymbol{\xi}}_n)]\Big\}^2 - \left[\frac{1}{n}\,\mathrm{LR}_n(\hat{\boldsymbol{\theta}}_n,\hat{\boldsymbol{\xi}}_n)\right]^2.$$

The GLR statistic is a consistent estimator of the unknown quantity D_g^{sup}. Under certain general regularity conditions, are obtained the following results on the asymptotic distribution of GLR_n (VUONG, 1989, Theorem 5.1):

i) under $\mathcal{H}_0$, $\ \mathrm{GLR}_n \overset{a}{\sim} \mathcal{N}(0,1)$,

ii) under $\mathcal{H}_{g^{sup}}$, $\ \mathrm{GLR}_n \xrightarrow{\text{a.s.}} +\infty$,

iii) under $\mathcal{H}_g$, $\text{GLR}_n \xrightarrow{\text{a.s.}} -\infty$,

where $\overset{a}{\sim}$ denotes asymptotic distribution and $\xrightarrow{\text{a.s.}}$ means convergence "almost surely".

These results provide simple tests to select between the models $G^{sup}_{\boldsymbol{\theta}}$ and $G_{\boldsymbol{\xi}}$. Certainly, let $z_{\alpha/2}$ be the $1-\alpha/2$ quantile of the standard normal distribution, for some significance level α. If $\text{GLR}_n > z_{\alpha/2}$, we reject the null hypothesis that the models are equivalent in favor of $G^{sup}_{\boldsymbol{\theta}}$ being better than $G_{\boldsymbol{\xi}}$. If $\text{GLR}_n < -z_{\alpha/2}$, we reject $\mathcal{H}_0$ in favor of $G_{\boldsymbol{\xi}}$ being better than $G^{sup}_{\boldsymbol{\theta}}$. Finally, if $|\text{GLR}_n| \leq z_{\alpha/2}$, then we can not discriminate between $G^{sup}_{\boldsymbol{\theta}}$ and $G_{\boldsymbol{\xi}}$ given the data. Similar results are obtained to considering the G^{inf} family.

2.9 Simulation study

In this section, we consider as special case the Fr^{sup} distribution (see Section 2.4.2) and perform a Monte Carlo simulation experiment in order to evaluate the behavior of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{\alpha}_n, \hat{\beta}_n, \hat{b}_n)$ in finite samples. We estimate the relative bias and mean squared error (MSE) for the sample sizes $n = 50, 100, 200$ and 300 and vary α and b in the set $\{0.5, 0.75\}$ and β in $\{1.5, 1.75\}$. We consider $10{,}000$ Monte Carlo replications and use the BFGS method with analytical derivatives for maximizing the log-likelihood function (2.27). All computations are performed using the `C` programming language and the `GNU` Scientific Library (version 2.1). The random variates are generated from the Fr^{sup} model according to Section 2.7.

The results, given in Table 2.1, reveal that, in general, the relative bias and MSE values decrease when n increases, which is to be expected since the MLE's are asymptotically unbiased. The values in this table also reveal that the relative bias and MSE for $\hat{b}_n$ decreases as the value of β increases. We can also note that the relative bias and MSE do not exceed, in absolute value, 0.20 and 0.13, respectively. Further, it can be noted in Table 2.1 that the parameter b was underestimated in some cases (negative relative bias).

The Figures 2.11 and 2.12 display the plots of the relative bias values and the Figures 2.13 and 2.14 display the plots of the MSE values considering the scenarios $\alpha = 0.5, \beta = 1.5, b = 0.5$ and $\alpha = 0.75, \beta = 1.75, b = 0.75$, respectively.

2.10 Applications

In this section, the potentiality of the G^{inf} and G^{sup} families is proved empirically by means of two applications.

Table 2.1: Relative bias and MSE values of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{\alpha}_n,\hat{\beta}_n,\hat{b}_n)$ for the Fr^{sup} model

				relative bias			MSE		
α	β	b	n	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\hat{b}_n$	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\hat{b}_n$
0.5	1.5	0.5	50	0.058	0.060	0.172	0.027	0.058	0.125
			100	0.053	0.042	0.150	0.022	0.035	0.110
			200	0.040	0.030	0.134	0.016	0.024	0.094
			300	0.034	0.024	0.119	0.014	0.019	0.083
		0.75	50	0.100	0.075	−0.036	0.023	0.062	0.108
			100	0.089	0.058	−0.030	0.019	0.038	0.099
			200	0.077	0.046	−0.028	0.016	0.025	0.088
			300	0.069	0.040	−0.023	0.013	0.020	0.082
	1.75	0.5	50	0.056	0.061	0.145	0.020	0.078	0.121
			100	0.048	0.043	0.134	0.016	0.047	0.108
			200	0.033	0.030	0.128	0.012	0.032	0.093
			300	0.028	0.025	0.113	0.010	0.026	0.082
		0.75	50	0.090	0.075	−0.051	0.017	0.086	0.110
			100	0.078	0.058	−0.040	0.014	0.052	0.100
			200	0.065	0.046	−0.033	0.011	0.034	0.089
			300	0.057	0.040	−0.025	0.009	0.027	0.081
0.75	1.5	0.5	50	0.047	0.059	0.196	0.058	0.057	0.127
			100	0.049	0.042	0.161	0.049	0.035	0.111
			200	0.038	0.029	0.139	0.036	0.024	0.094
			300	0.033	0.024	0.122	0.030	0.019	0.083
		0.75	50	0.090	0.077	−0.022	0.050	0.063	0.106
			100	0.085	0.058	−0.024	0.042	0.037	0.098
			200	0.076	0.046	−0.026	0.035	0.025	0.088
			300	0.068	0.040	−0.022	0.030	0.020	0.082
	1.75	0.5	50	0.035	0.057	0.196	0.041	0.075	0.126
			100	0.036	0.041	0.163	0.034	0.046	0.111
			200	0.028	0.029	0.140	0.026	0.031	0.094
			300	0.025	0.024	0.121	0.022	0.026	0.083
		0.75	50	0.076	0.076	−0.028	0.034	0.084	0.106
			100	0.070	0.058	−0.026	0.029	0.051	0.098
			200	0.062	0.046	−0.026	0.024	0.034	0.088
			300	0.055	0.039	−0.021	0.021	0.026	0.081

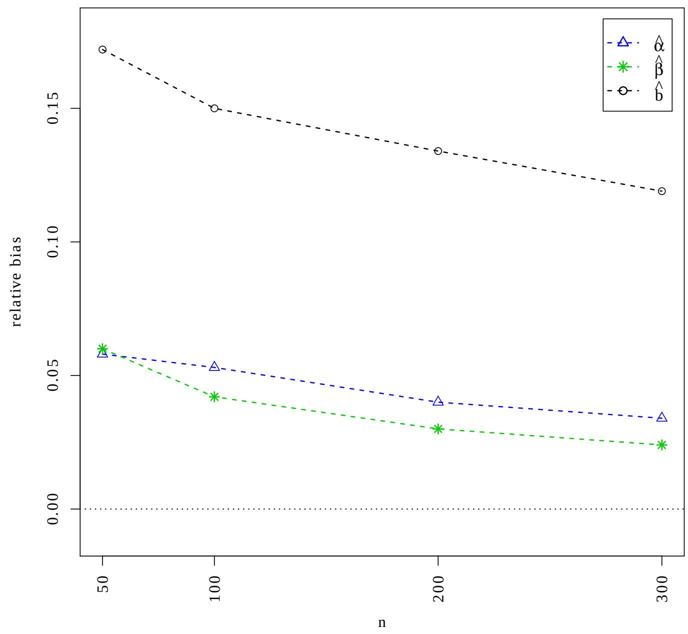


Figure 2.11: Relative bias for the parameter values $\alpha = 0.5, \beta = 1.5$ and $b = 0.5$ in the Fr^{sup} model.

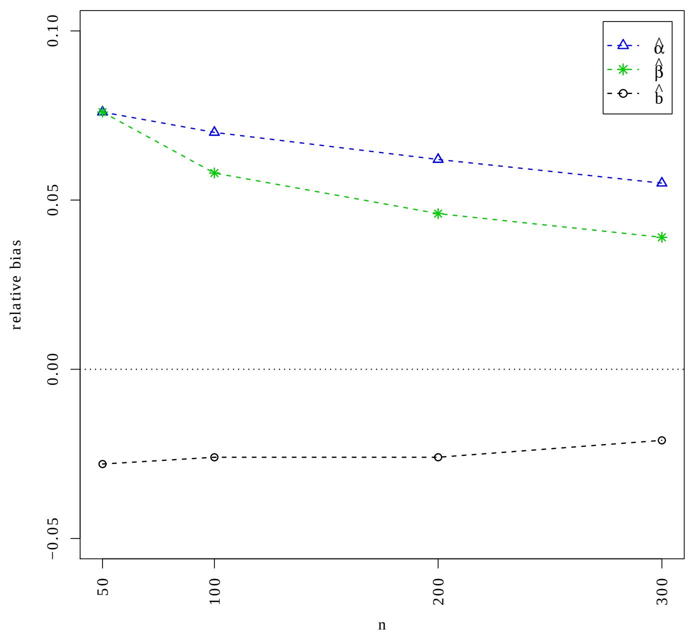


Figure 2.12: Relative bias for the parameter values $\alpha = 0.75, \beta = 1.75$ and $b = 0.75$ in the Fr^{sup} model.

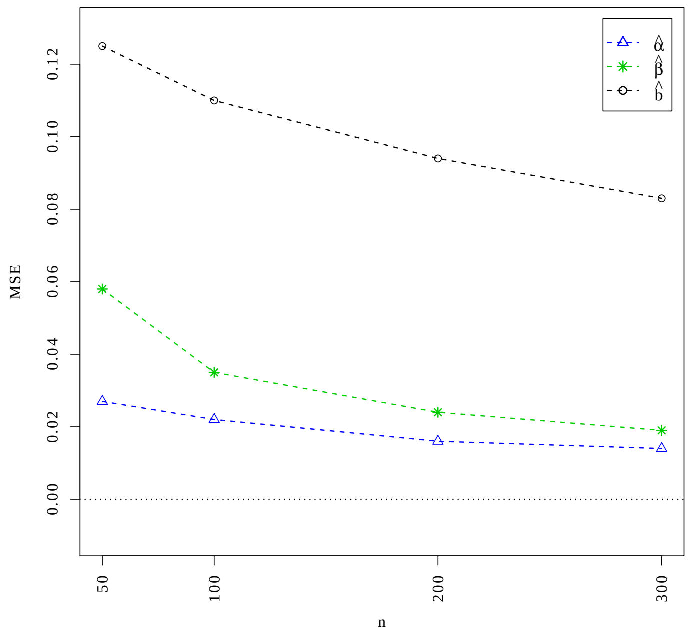


Figure 2.13: MSE for the parameter values $\alpha = 0.5, \beta = 1.5$ and $b = 0.5$ in the Fr^{sup} model.

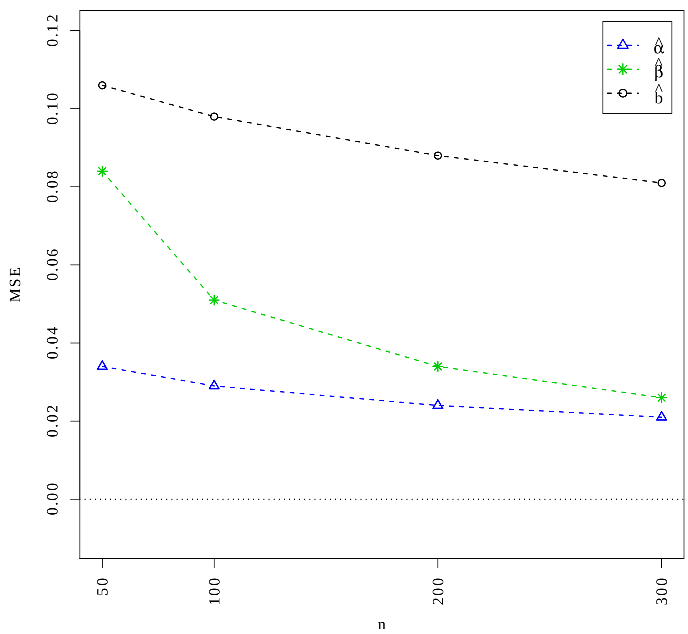


Figure 2.14: MSE for the parameter values $\alpha = 0.75, \beta = 1.75$ and $b = 0.75$ in the Fr^{sup} model.

a)*Traffic data*: As a first application, we consider a data set given in Jørgensen (1982) corresponding to the length of 128 intervals between the times at which vehicles pass a point on a road. The descriptive statistics for this data set are given in Table 2.2. To adjust the data, we consider the Lindley (Li), log-logistic (LL), Lomax (Lo), Fréchet (Fr), Weibull (W), LL^{inf}, Lo^{inf} and Fr^{sup} distributions. All computations are performed using the `R` software (version 3.0.2, `AdequacyModel` package).

For maximizing the log-likelihood function, we use the BFGS method with numerical derivatives. For purposes of comparison, we compute some goodness-of-fit statistics: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Cramér-von Mises Criterion (W*) and Anderson-Darling Criterion (A*) (CHEN; BALAKRISHNAN, 1995). In general, small values of these statistics suggest a better fit. The MLE's are given in Table 2.3 with their standard errors in parentheses and the goodness-of-fit values for the fitted distributions are listed in Table 2.4.

The figures in Table 2.4 reveal that the Fr^{sup} distribution has the smallest values of all statistics among the fitted models. Therefore, based on these statistics, we can conclude that the Fr^{sup} distribution gives the best fit to the current data. The plots in Figure 2.15 display the Fréchet, LL^{inf} and Fr^{sup} estimated densities.

b)*Strength data*: The second application, which was originally reported by Bader & Priest (1982), represents the strength measured in GPa for 63 single carbon fibers tested under tension at gauge lengths of 10 mm. These data also are used by Kundu & Raqab (2009). The descriptive statistics for this data set are given in Table 2.2. In this case, we consider the Lindley (Li), log-logistic (LL), Fréchet (Fr), Weibull (W), LL^{inf} and Fr^{sup} distributions. The MLE's are given in Table 2.5 with their standard errors in parentheses and the goodness-of-fit values for the fitted distributions are listed in Table 2.6. The figures in Table 2.6 reveal that the Fr^{sup} distribution has the smallest values of the AIC, HQIC, W* and A* statistics among the fitted models. So, these statistics suggest that the Fr^{sup} distribution gives the best fit to the current data. The plots in Figure 2.16 display the LL, LL^{inf} and Fr^{sup} estimated densities.

To analyze how significant is the additional parameter b of the Fr^{sup} distribution for modeling the current data, we use the GLR statistic, as discussed in Section 2.8.1, for testing the Fréchet model versus the Fr^{sup} model. The results are given in Table 2.7 for both data sets. If $z_{\alpha/2} \approx 2.576$ is the $1-\alpha/2$ quantile of the standard normal distribution at the 1% significance level, from Table 2.7 we have $\text{GLR}_n > z_{\alpha/2}$ in both applications. Thus, we reject the null hypoteses that the Fréchet and Fr^{sup} models are equivalent in favor of the Fr^{sup} model being better than the Fréchet model. So, we have evidence of the potencial need for including the parameter b to model both data sets.

Table 2.2: Descriptive statistics for the data sets

Application	min.	1st quantile	median	mean	3rd quantile	max.
Traffic data	0.200	1.975	5.850	15.810	16.550	125.300
Strength data	1.901	2.554	2.996	3.059	3.422	5.020

Table 2.3: MLE's and standard errors for the traffic data

	MLE		
Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{b}$
Li(α)	0.120 (0.008)	-	-
LL(α,β)	6.048 (0.769)	1.236 (0.089)	-
Lo(α,β)	1.610 (0.393)	12.005 (4.426)	-
Fr(α,β)	3.314 (0.378)	0.818 (0.054)	-
W(α,β)	12.847 (1.615)	0.746 (0.049)	-
LL^{inf}(α,β,b)	5.752 (0.809)	1.152 (0.090)	7.757 (2.922)
Lo^{inf}(α,β,b)	1.448 (0.354)	9.759 (3.831)	6.892 (2.378)
Fr^{sup}(α,β,b)	3.449 (0.427)	0.778 (0.059)	9.862 (4.741)

Table 2.4: Goodness-of-fit statistics for the traffic data

	Statistic				
Distribution	AIC	BIC	HQIC	W*	A*
Li($\hat{\alpha}$)	1062.24	1065.09	1063.40	0.791	4.738
LL($\hat{\alpha}$,$\hat{\beta}$)	930.01	935.71	932.32	0.326	1.976
Lo($\hat{\alpha}$,$\hat{\beta}$)	933.30	939.00	935.62	0.360	2.208
Fr($\hat{\alpha}$,$\hat{\beta}$)	925.16	930.86	927.47	0.159	1.043
W($\hat{\alpha}$,$\hat{\beta}$)	943.38	949.09	945.70	0.526	3.206
LL^{inf}($\hat{\alpha}$,$\hat{\beta}$,$\hat{b}$)	922.24	930.79	925.71	0.215	1.289
Lo^{inf}($\hat{\alpha}$,$\hat{\beta}$,$\hat{b}$)	922.55	931.11	926.03	0.231	1.394
Fr^{sup}($\hat{\alpha}$,$\hat{\beta}$,$\hat{b}$)	**918.88**	**927.44**	**922.36**	**0.121**	**0.788**

Table 2.5: MLE's and standard errors for the strength data

	MLE		
Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{b}$
Li(α)	0.539 (0.050)	-	-
LL(α,β)	2.993 (0.077)	8.646 (0.890)	-
Fr(α,β)	2.721 (0.067)	5.434 (0.508)	-
W(α,β)	3.315 (0.088)	5.049 (0.456)	-
LL^{inf}(α,β,b)	2.981 (0.082)	8.139 (0.899)	11.463 (7.319)
Fr^{sup}(α,β,b)	2.758 (0.081)	4.955 (0.576)	5.653 (3.368)

Table 2.6: Goodness-of-fit statistics for the strength data

Distribution	Statistic				
	AIC	BIC	HQIC	W*	A*
Li($\hat{\alpha}$)	244.72	246.86	245.56	0.060	0.378
LL($\hat{\alpha}$,$\hat{\beta}$)	119.69	**123.98**	121.38	0.089	0.502
Fr($\hat{\alpha}$,$\hat{\beta}$)	121.80	126.09	123.49	0.115	0.642
W($\hat{\alpha}$,$\hat{\beta}$)	127.91	132.20	129.60	0.128	0.892
LL^{inf}($\hat{\alpha}$,$\hat{\beta}$,$\hat{b}$)	119.07	125.50	121.59	0.069	0.362
Fr^{sup}($\hat{\alpha}$,$\hat{\beta}$,$\hat{b}$)	**118.55**	124.98	**121.08**	**0.055**	**0.301**

Table 2.7: Generalized LR tests

Hypotheses	Application	GLR statistic	p-value
$D_g^{sup} = 0$ vs $D_g^{sup} \neq 0$	Traffic data	5.339	4.67×10^{-8}
	Strength data	3.928	4.28×10^{-5}

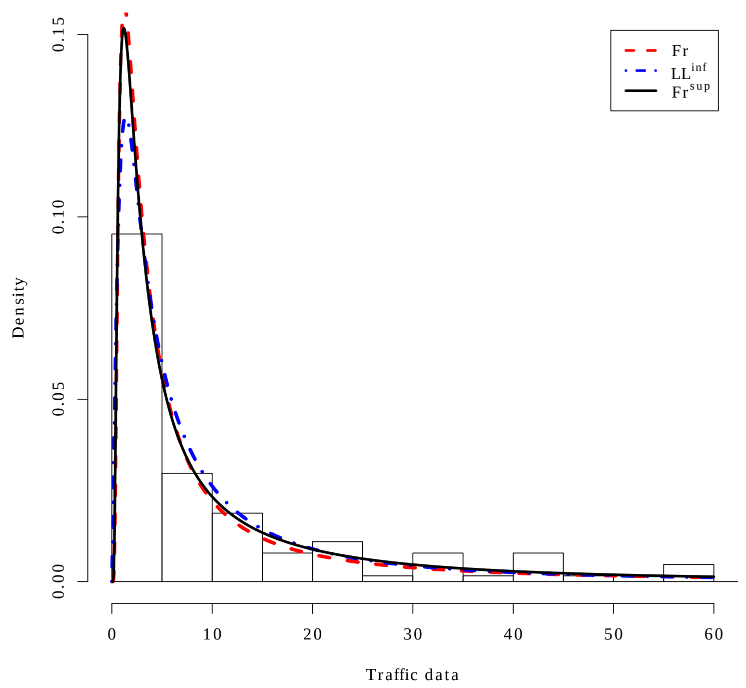


Figure 2.15: The Fréchet, LL^{inf} and Fr^{sup} estimated densities for the traffic data.

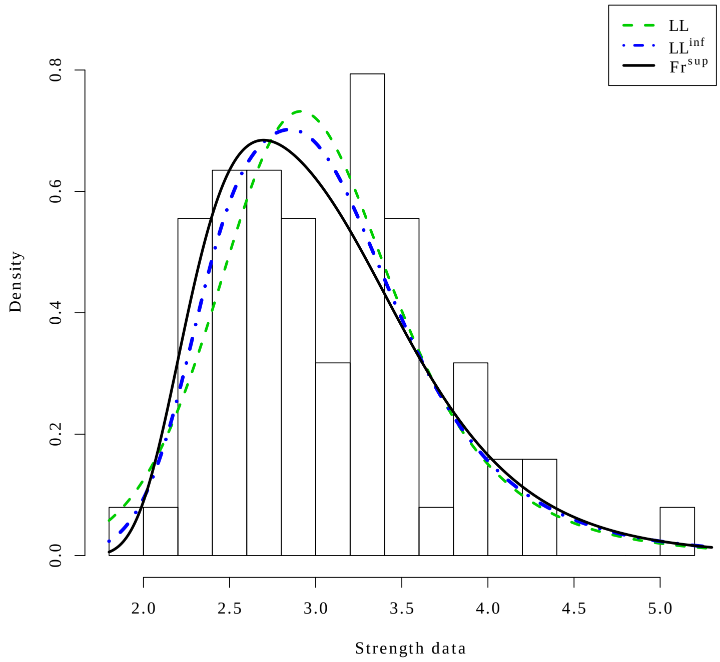


Figure 2.16: The LL, LL^{inf} and Fr^{sup} estimated densities for the strength data.

2.11 Conclusion and final remarks

In this chapter, we introduce two new generalized G families, named the supremum and infimum families of distributions, to inducing one additional shape parameter to the parent distribution G. We present some motivations to introduce these families, give a physical interpretation and prove these families have simple expressions for moments in terms of moments of exponentiated-G (exp-G) distributions. Several special lifetime distributions belonging to both families are present and they reveal that the supremum family can induce bathtub shape in its hazard rate function (hrf). Asymptotics and shapes of the new families are given and we obtain some structural properties of these families. Also, we provide a general method for obtain random variates from the supremum and infimum families. In addition, maximum likelihood estimates for complete samples from these families are also considered and we perform a Monte Carlo simulation in order to evaluate the behavior of these estimates for the Fréchet supremum (Fr^{sup}) model. Finally, we compare the performance of some new special lifetime distributions with other lifetime distributions by using the classical goodness-of-fit statistics. The results confirm that the distributions belonging to the supremum and infimum families are very appropriate for lifetime applications.

Chapter 3

The modified Fréchet distribution and its properties

Resumo

A distribuição Fréchet é um modelo absolutamente contínuo que tem ampla aplicabilidade na teoria de valores extremos. Contudo, esse modelo apresenta um desempenho limitado e não pode representar muitas das situações práticas. Assim, neste capítulo propomos uma nova distribuição de três parâmetros, denominada Fréchet modificada, obtida ao extender a distribuição Fréchet. Diferentemente do modelo Fréchet, a nova distribuição apresenta função razão de risco decrescente e unimodal invertida, a qual fornece interpretações muito intuitivas em diversas situações. Obtemos algumas propriedades da nova distribuição ao fazer uso da função de Lambert. Consideramos um estudo de simulação para ilustrar o desempenho das estimativas de máxima verossimilhança. A flexibilidade da distribuição introduzida é ilustrada por meio de um conjunto de dados reais. Fazemos uso de algumas estatísticas de bondade de ajuste para verificar a adequabilidade do novo modelo, provando assim que ele pode ser apropriado para aplicações a dados reais.

Palavras-chave: Análise de tempo de vida, distribuição Fréchet, distribuição Fréchet modificada, função W de Lambert.

Abstract

The Fréchet distribution is an absolutely continuous model which has wide applicability in extreme value theory. However, this model have a limited range of behaviour and can not represent all the situations found in applications. Thus, in this chapter, we propose a new three-parameter model, so-called the modified Fréchet distribution, to extend the Fréchet distribution. Differently from the Fréchet model, the new distribution allows decreasing and inverted unimodal hazard rate, which have considerable intuitive appeal. By using the Lambert function, we obtain some properties of the new distribution. We provide a simulation study to illustrate the performance of the maximum likelihood estimates. The flexibility of the introduced distribution is illustrated by means of a real data set. We use some goodness-of-fit statistics to verify the adequacy of the proposed model. We prove empirically that it can be appropriated for applications to real data sets.

Keywords: Fréchet distribution, Lambert W function, lifetime analysis, modified Fréchet distribution.

3.1 Introduction

The type II extreme value distribution, also known as the Fréchet distribution, is a family of continuous distributions developed within the general extreme value theory, which deals with the stochastic behaviour of the maximum and the minimum of independent and identically distributed (i.i.d.) random variables (KOTZ; NADARAJAH, 2000). This distribution was introduced by Maurice Fréchet (1878-1973), who investigated it as one possible limit distribution for a sequence of maxima of i.i.d. random variables.

Although the Fréchet distribution is used in applications involving stochastic phenomena such as rainfall, floods, air pollution (KOTZ; NADARAJAH, 2000) or material properties in engineering applications (HARLOW, 2002), this model have a limited range of behaviour and can not represent all the situations found in applications. For example, this model does not allow bathtub hazard rate, which is widely used in lifetime applications by its considerable intuitive appeal.

With the aim of obtaining more flexibility, several extensions of the Fréchet distribution were proposed in the literature. Some of them are: the Kumaraswamy Fréchet (MEAD, 2014), beta Fréchet (NADARAJAH; GUPTA, 2004; BARRETO SOUZA *et al.*, 2011), exponentiated Fréchet (NADARAJAH; KOTZ, 2001), Marshall-Olkin Fréchet (KRISHNA *et al.*, 2013), transmuted Fréchet (MAHMOUD; MANDOUH, 2013), gamma extended Fréchet (SILVA *et al.*, 2013), transmuted exponentiated Fréchet (ELBATAL *et al.*, 2014), transmuted Marshall-Olkin Fréchet (AFIFY *et al.*, 2015) and transmuted exponentiated generalized Fréchet (YOUSOF *et al.*, 2015) distributions.

In this chapter, we propose a new three-parameter extended Fréchet model named the *modified Fréchet* (MF) distribution. Differently from the Fréchet model, the new distribution allows decreasing and inverted unimodal hazard rate, which are useful for modeling various cases in lifetime applications (LAI, 2013; MARSHALL; OLKIN, 1997). So, our aim is to define a more flexible model for lifetime applications. In addition, given its simple mathematical form, several of its mathematical quantities and properties such as the quantile function (qf) and expansions for the ordinary moments are obtained based on the Lambert W function (CORLESS *et al.*, 1996; CARRASCO *et al.*, 2008; SILVA *et al.*, 2010; JODRÁ, 2010).

The chapter unfolds as follows. In Section 3.2, we review some main issues of the Lambert W function. In Section 3.3, we introduce the MF model and discuss the generation of random variates from this distribution by considering the Lambert function. In Section 3.4, we plot its density and hazard rate functions for some parameter values. In Section 3.5, we express the qf in terms of the Lambert function. In Sections 3.6 and 3.7, we present expansions for the ordinary and incomplete moments, generating function and Bonferroni and Lorenz curves using an expansion for the Lambert function. In Section 3.8, we determine the order statistics and their moments. Section 3.9 is devoted to the maximum likelihood estimates (MLE's) for complete samples and, in Section 3.10, we carry out a simulation study to evaluate the performance of these estimates. In Section 3.11, we provide an application of the MF distribution and compare it with the Fréchet distribution and others three-parameter extended distributions including the exponentiated Weibull distribution (MUDHOLKAR; SRIVASTAVA, 1993), since it is a widely used lifetime model. Finally, Section 3.12 concludes the chapter.

3.2 The Lambert W function

The Lambert W function (CORLESS *et al.*, 1996; JODRÁ, 2010) has been applied to solve several problems in mathematics, physics and engineering. It is implicitly defined as the branches of the inverse relation of the function $\tau(z) = z\,\mathrm{e}^{z}$, $z \in \mathbb{C}$, that is

$$z = \tau^{-1}(z\,\mathrm{e}^{z}) = W(z\,\mathrm{e}^{z}), \quad z \in \mathbb{C}.$$

The Lambert function can not be expressed in terms of elementary functions. However, a feature that makes the Lambert function attractive is that it is an analytic function, that is, this function is locally given by a convergent power series. Figure 3.1 displays plots of $W(x)$ for $x \in \mathbb{R}$. We note that for $-\mathrm{e}^{-1} \leq x < 0$, there are two possible real values of $W(x)$ and, therefore, two branches are defined: the branch satisfying

$W(x) \geq -1$ is denoted by $W_0(x)$ and called the principal branch of $W(\cdot)$, and the other branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$ and called the negative branch of $W(\cdot)$.

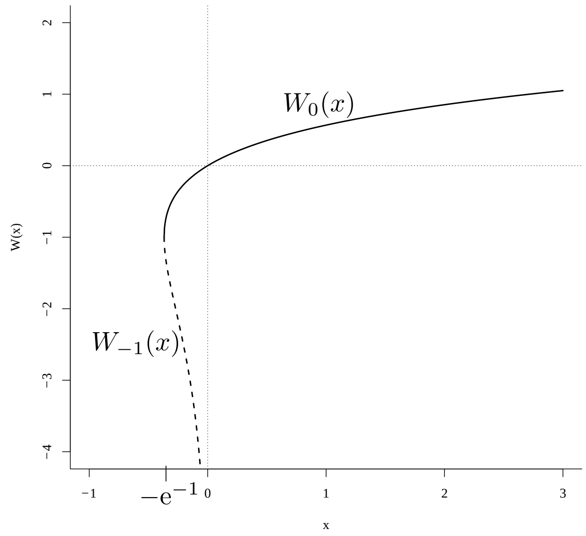


Figure 3.1: Real branches of the Lambert W function.

Next, we present some properties and expansions for $W_0(\cdot)$ (CORLESS *et al.*, 1996). By definition of the Lambert function, the principal branch satisfies

$$x = W_0(x\,\mathrm{e}^x), \quad x \geq -1. \tag{3.1}$$

From equation (3.1), we obtain (for $x \geq -1$)

$$W_0(x\,\mathrm{e}^x)\,\mathrm{e}^{W_0(x\,\mathrm{e}^x)} = x\,\mathrm{e}^x,$$

and, therefore, letting $y = x\,\mathrm{e}^x$, we note that $W_0(\cdot)$ is the solution of the equation

$$W_0(y)\,\mathrm{e}^{W_0(y)} = y, \quad y \geq -\mathrm{e}^{-1}. \tag{3.2}$$

Furthermore, for $|x| \leq \mathrm{e}^{-1}$ and $r \in \mathbb{N}$, the function $W_0^r(x)$ has a power series around $x_0 = 0$ given by

$$W_0^r(x) = \sum_{n=r}^{\infty} \frac{r\,(-n)^{n-r}}{n\,(n-r)!}\,x^n. \tag{3.3}$$

Other useful expansion related $W_0(\cdot)$ with the exponential function is given by

$$\mathrm{e}^{c\,W_0(x)} = \sum_{n=0}^{\infty} \frac{c\,(c-n)^{n-1}}{n!}\,x^n, \quad c \in \mathbb{C},\, |x| \leq \mathrm{e}^{-1}. \tag{3.4}$$

The analytical properties of the function $W_0(x)$ presented above will be used henceforth to obtain several mathematical quantities and properties of the MF distribution, which is introduced in the following section. Since the Lambert W function is implemented in various scientific libraries, computations of quantities such as quantiles and random variates related to the MF distribution can be implemented more efficiently.

3.3 The new distribution

The Fréchet model is a special case of the generalized extreme value distribution, which is a family of continuous distributions that includes as special cases the Gumbel, Fréchet and Weibull distributions, also known as type I, type II and type III extreme value distributions, respectively (KOTZ; NADARAJAH, 2000; ALVES; NEVES, 2011). Its cumulative distribution function (cdf) and probability density function (pdf) are given by

$$G(x;\alpha,\beta) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right], \quad x > 0, \quad \alpha,\beta > 0, \tag{3.5}$$

and

$$g(x;\alpha,\beta) = \frac{\beta}{x}\left(\frac{\alpha}{x}\right)^{\beta} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right].$$

Generalizing distributions is an old practice and has ever been considered as precious as other practical problems in Statistics. The modern era on distribution theory stresses on problem-solving faced by the applied researchers to propose a variety of models so that a data can be better assessed and explored that are available in different fields of life. In other words, there is strong need to introduce useful models for better exploration of the real-life phenomenons.

Lai *et al.* (2003) have successfully defined the three-parameter modified Weibull distribution by taking appropriate limits on the beta integrated distributions. Also, it is obtained by extending the Weibull distribution by incluing the additional term $\mathrm{e}^{\lambda x}$ in order to decrease more rapidly the survival function. Following a similar approach, in this chapter we study the MF distribution by extending the cdf (3.5) to the form

$$F(x;\alpha,\beta,\lambda) = \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\lambda x}\right], \quad x>0, \quad \alpha,\beta>0, \quad \lambda \geq 0, \tag{3.6}$$

where α, β and λ are shape parameters. It is straightforward matter to observe that $F(x)$ is differentiable and strictly increasing in $(0,\infty)$ and that $\lim_{x\to 0} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. Therefore, $F(x)$ is, in fact, a legitimate absolutely continuous cdf. Thus, the corresponding MF density is given by

$$f(x;\alpha,\beta,\lambda) = \frac{1}{x}\,(\beta+\lambda x)\left(\frac{\alpha}{x}\right)^{\beta} \exp\left[-\lambda x - \left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\lambda x}\right]. \tag{3.7}$$

Note that the Fréchet distribution is a special case of the MF distribution when $\lambda = 0$. Hereafter, a random variable X with pdf (3.7) will be denoted by $X \sim \mathrm{MF}(\alpha,\beta,\lambda)$.

In lifetime analysis, a very useful function is the hazard rate function (hrf) $r(x) = f(x)/[1-F(x)]$ (MARSHALL; OLKIN, 2007). The hrf of X is given by

$$r(x;\alpha,\beta,\lambda) = \frac{(\beta+\lambda\, x)\,(\frac{\alpha}{x})^{\beta}\,\mathrm{e}^{-\lambda x}}{x\left\{\exp\left[(\frac{\alpha}{x})^{\beta}\,\mathrm{e}^{-\lambda x}\right]-1\right\}}. \tag{3.8}$$

3.3.1 Generating random variates

Inverting (3.6), a random variable X with pdf (3.7) can be simulated as follows. Let u be an observation of the random variable $U \sim \mathcal{U}(0,1)$. Then, an observation of X can be obtained as a solution of the nonlinear equation

$$\left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\lambda x} + \log(u) = 0. \tag{3.9}$$

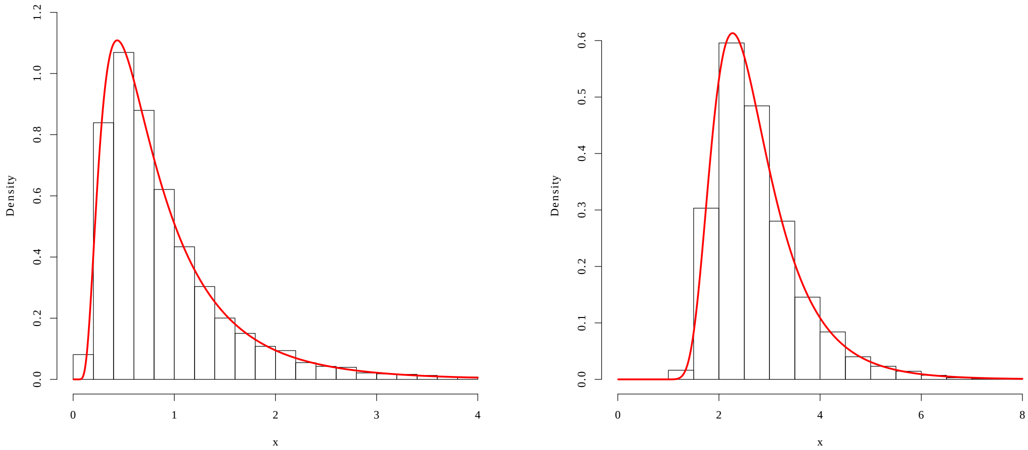
Numerical algorithms, such as Newton-Raphson methods, can be used for determining x from (3.9).

It is possible to go a step further in (3.9) and simulate X in a form computationally more efficient by using the Lambert W function (see Section 3.2). From (3.9) and after some algebraic manipulation, we can write

$$\frac{\lambda\, x}{\beta}\,\mathrm{e}^{\frac{\lambda x}{\beta}} = \frac{\alpha\,\lambda}{\beta\left[\log\left(\frac{1}{u}\right)\right]^{1/\beta}}.$$

Applying $W_0(\cdot)$ in both sides and using (3.1) gives

$$\frac{\lambda\, x}{\beta} = W_0\left(\frac{\alpha\,\lambda}{\beta\left[\log\left(\frac{1}{u}\right)\right]^{1/\beta}}\right).$$



(a) $\alpha = 1.0$, $\beta = 1.0$, $\lambda = 1.0$ (b) $\alpha = 6.0$, $\beta = 2.0$, $\lambda = 0.8$

Figure 3.2: Plots of the exact MF densities and histograms of the simulated data for given parameters.

Thus, we obtain the following result: if $U \sim \mathcal{U}(0,1)$, then

$$X = Q(U) = \frac{\beta}{\lambda} W_0 \left(\frac{\alpha \lambda}{\beta \left[\log \left(\frac{1}{U} \right) \right]^{1/\beta}} \right) \sim \mathrm{MF}(\alpha,\beta,\lambda). \tag{3.10}$$

In Figure 3.2, we compare the exact MF densities and histograms from two simulated data sets for specified parameters by showing the consistent of the simulated values from (3.10) with the MF distribution. To simulate the data, we use the `R` software (version 3.0.2, `lamW` package).

3.4 Shapes of the density function

The shapes of the pdf (3.7) can be described analytically by examining the roots of the equation $f'(x) = 0$ and analyzing its limits when $x \to 0$ or $x \to \infty$. Clearly, since $f(x) \geq 0$ is integrable, then $\lim_{x \to \infty} f(x) = 0$. The following result gives the limit of $f(x)$ when $x \to 0$.

PROPOSITION 2 $\lim_{x \to 0} f(x) = 0$.

Proof:

From (3.7) we note that $f(x)$ can be expressed as

$$f(x) = \left(\lambda + \frac{\beta}{x} \right) \left(\frac{\alpha}{x} \right)^{\beta} \mathrm{e}^{-\lambda x} \exp\left[-\left(\frac{\alpha}{x} \right)^{\beta} \mathrm{e}^{-\lambda x} \right].$$

Since $x > 0$ and $\lambda \geq 0$, then $\mathrm{e}^{-\lambda x} \leq 1$ and, therefore,

$$0 \leq f(x) \leq \left(\lambda + \frac{\beta}{x}\right)\left(\frac{\alpha}{x}\right)^{\beta} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\lambda x}\right].$$

Using the fact that $\lim\limits_{x \to 0} \mathrm{e}^{-\lambda x} = 1$, we have

$$0 \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0} \left(\lambda + \frac{\beta}{x}\right)\left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\beta}}. \tag{3.11}$$

Next, we will prove that

$$\lim_{x \to 0} \frac{\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{x^{\beta+1}} = 0. \tag{3.12}$$

Indeed, we have the known result for all $n \in \mathbb{N}$, $\lim\limits_{x \to \infty} x^n \mathrm{e}^{-x} = 0$. Letting $y = \left(\frac{\alpha}{x}\right)^{\beta}$, we have $x = \frac{\alpha}{y^{1/\beta}}$. Thus, $x \to 0$ if and only if $y \to \infty$ and, therefore,

$$\lim_{x \to 0} \frac{\mathrm{e}^{-\left(\frac{\alpha}{x}\right)^{\beta}}}{x^{\beta+1}} = \lim_{y \to \infty} \frac{y^{\frac{\beta+1}{\beta}} \mathrm{e}^{-y}}{\alpha^{\beta+1}}.$$

Let $n \in \mathbb{N}$ such that $\frac{\beta+1}{\beta} \leq n$. Then, for $y \geq 1$, we have $y^{\frac{\beta+1}{\beta}} \leq y^n$ and then

$$0 \leq \lim_{y \to \infty} \frac{y^{\frac{\beta+1}{\beta}} \mathrm{e}^{-y}}{\alpha^{\beta+1}} \leq \lim_{y \to \infty} \frac{y^n \mathrm{e}^{-y}}{\alpha^{\beta+1}} = 0,$$

which proves (3.12). Finally, using (3.12) in the inequality (3.11), we obtain $\lim\limits_{x \to 0} f(x) = 0$. ■

Since $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to \infty} f(x) = 0$, then $f(x)$ must have at least one mode. The following result shows that, in fact, $f(x)$ is unimodal.

PROPOSITION 3 *$f(x)$ is unimodal, with mode $x = x_0$ satisfying*

$$\left(\frac{\alpha}{x_0}\right)^{\beta} \mathrm{e}^{-\lambda x_0} - \frac{\beta}{(\beta + \lambda x_0)^2} - 1 = 0. \tag{3.13}$$

Sketch of the Proof:

For $\lambda = 0$, we obtain as particular case the Fréchet distribution, which is well known to be unimodal. Suppose, then, $\lambda > 0$. Let $s(x) = u(x) + v(x)$ be, where

$$u(x) = \left(\frac{\alpha}{x}\right)^{\beta} (\beta + \lambda x)^2 \text{ and } v(x) = -\mathrm{e}^{\lambda x}\left[\beta + (\beta + \lambda x)^2\right].$$

It is easy see that $u(x) > 0$, $v(x) < 0$ and $v(x)$ is strictly decreasing, for all $x > 0$ and $\alpha, \beta, \lambda > 0$. Furthermore, $\lim_{x\to 0} u(x) = \infty$, $\lim_{x\to 0} v(x) = -\beta(1+\beta)$ and $\lim_{x\to\infty} v(x) = -\infty$. So, $\lim_{x\to 0} s(x) = \infty$.

To calculate $f'(x)$, we obtain

$$f'(x) = \frac{1}{x^2}\left(\frac{\alpha}{x}\right)^{\beta} \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\lambda x} - 2\lambda x\right] s(x)$$

and, therefore, $f'(x) = 0$ if and only if $s(x) = 0$. Since $f(x)$ must have at least one mode, there exists $x_0 \in (0,\infty)$ such that $s(x_0) = 0$.

We have

$$u'(x) = (\beta + \lambda x)\left(\frac{\alpha}{x}\right)^{\beta}\left(2\lambda - \frac{\beta(\beta+\lambda x)}{x}\right).$$

Consider two cases: $\beta \geq 2$ and $0 < \beta < 2$. An analysis of $u'(x)$ reveals that, in the first case, $u(x)$ is strictly decreasing in $(0,\infty)$, with $\lim_{x\to\infty} u(x) < \infty$. So, $s(x)$ is strictly decreasing in $(0,\infty)$ and $\lim_{x\to\infty} s(x) = -\infty$. In the second case, we have that $u(x)$ is strictly convex, with minimum point at $x^* = \frac{\beta^2}{(2-\beta)\lambda}$. Since $u(x)$ is strictly decreasing in $(0,x^*)$, then $s(x)$ is also strictly decreasing in this interval. To see that $s(x)$ is strictly decreasing in (x^*,∞), consider $x = x^* + z$, with $z > 0$. Thus, $x \in (x^*,\infty)$ and

$$s(x) = \tilde{s}(z) = \left(\frac{2\,\beta}{2-\beta} + \lambda\, z\right)^2 \left(\frac{\alpha(2-\beta)\lambda}{\beta^2 + (2-\beta)\lambda\, z}\right)^{\beta} - \mathrm{e}^{\beta^2/(2-\beta)}\mathrm{e}^{\lambda z}\left[\left(\frac{2\,\beta}{2-\beta} + \lambda\, z\right)^2 + \beta\right].$$

An analysis of $\tilde{s}(z)$, for $z > 0$, reveals that $\tilde{s}(z)$ is strictly decreasing and, because of the exponential term, we have $\lim_{z\to\infty} \tilde{s}(z) = -\infty$. We conclude then that, in the second case, $s(x)$ is strictly decreasing in $(0,\infty)$ and $\lim_{x\to\infty} s(x) = -\infty$, too.

Since $s(x_0) = 0$, $\lim_{x\to 0} s(x) = \infty$, $\lim_{x\to\infty} s(x) = -\infty$ and $s(x)$ is strictly decreasing in $(0,\infty)$, we conclude that x_0 is the unique point in $(0,\infty)$ such that $f'(x_0) = 0$. Thus, $f(x)$ is unimodal, with mode $x = x_0$ satisfying

$$\frac{\mathrm{e}^{-\lambda x_0}}{(\beta + \lambda\, x_0)^2}\, s(x_0) = \left(\frac{\alpha}{x_0}\right)^{\beta} \mathrm{e}^{-\lambda x_0} - \frac{\beta}{(\beta + \lambda x_0)^2} - 1 = 0.$$

■

The shape parameters α, β and λ allow extensive control on the right tail, providing more light or heavy tails, according β and λ decreases (α increases) or β and λ increases (α decreases), respectively.

The following corollary of the Proposition 3 gives the behavior of the mode of $f(x)$ in function of the parameters. Hereafter, the symbol $\to$ means "tends to".

COROLLARY 1 *Let x_0 the mode of $f(x)$. then*

a) $\alpha \to 0^+$ *implies that* $x_0 \to 0^+$ *and* $\alpha \to \infty$ *implies that* $x_0 \to \infty$.

b) $\beta \to 0^+$ *implies that* $x_0 \to 0^+$.

c) $\lambda \to 0^+$ *implies that* $x_0 \to \alpha\left(\frac{\beta}{\beta+1}\right)^{1/\beta}$.

Proof:

a) Note that, from equation (3.13), we have

$$\alpha = x_0\,\mathrm{e}^{\lambda x_0/\beta}\left[\frac{\beta}{(\beta+\lambda x_0)^2}+1\right]^{1/\beta}$$

Thus, $\alpha \to 0^+$ implies that $x_0 \to 0^+$ and $\alpha \to \infty$ implies that $x_0 \to \infty$.

b) From (3.13), we have that $\beta \to 0^+$ implies that $\mathrm{e}^{-\lambda x_0} \to 1$, which implies that $x_0 \to 0^+$.

c) From (3.13), we have that $\lambda \to 0^+$ implies that $\left(\frac{\alpha}{x_0}\right)^{\beta} - \frac{1}{\beta} - 1 \to 0$, which implies that $x_0 \to \alpha\left(\frac{\beta}{\beta+1}\right)^{1/\beta}$, the mode of the Fréchet distribution.

■

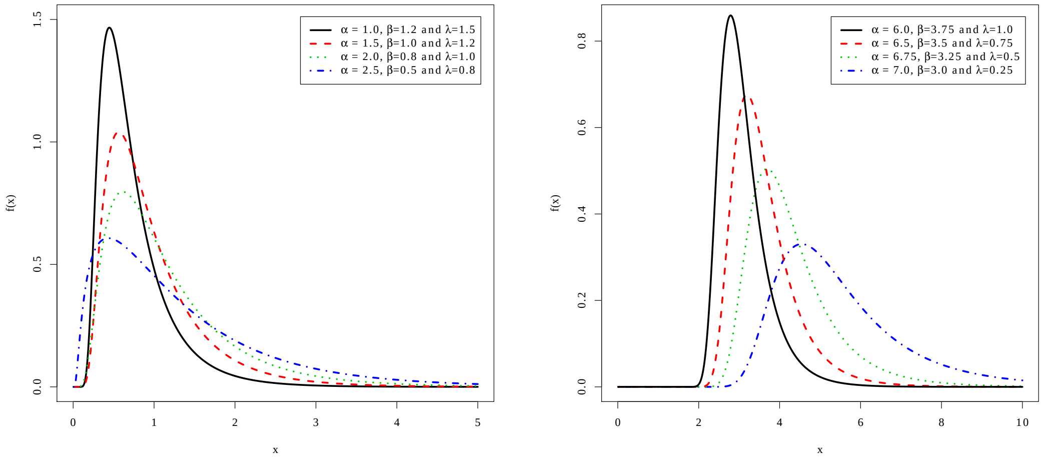
The plots in Figure 3.3 display the shapes of the pdf of X for some parameter values.

The plots of the hrf of X displayed in Figure 3.4 reveal the classical shapes such as decreasing, unimodal and inverted unimodal. So, the new distribution can be appropriate for different applications in lifetime analysis.

3.5 Quantile function

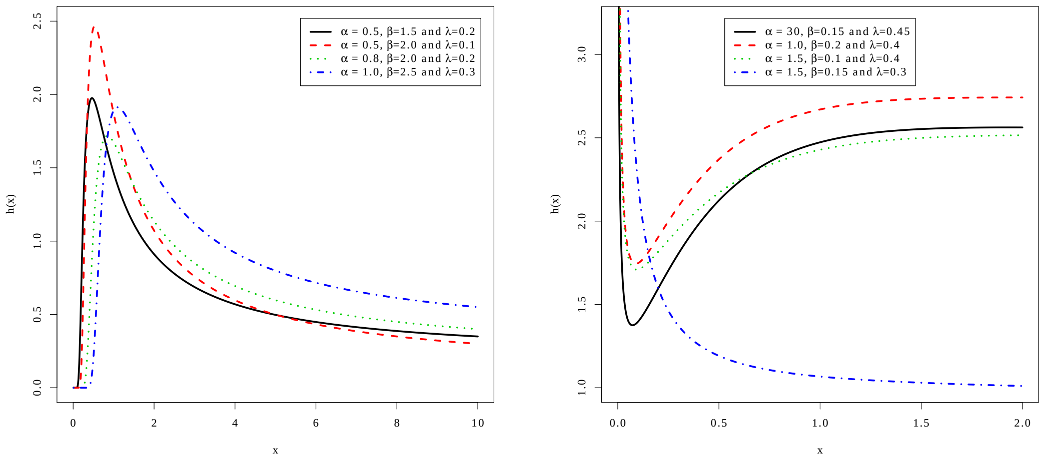
Since the cdf $F(x)$ given in (3.6) is continuous and strictly increasing, the qf of X is $Q(u) = F^{-1}(u)$, for $0 < u < 1$. From Section 3.3.1, we note that, for $\lambda > 0$, the qf of the MF distribution can be given explicitly in terms of the Lambert function as in (3.10).

Useful skewness and kurtosis measures are given by $\alpha_3 = \mu_3/\sigma^3$ and $\alpha_4 = \mu_4/\sigma^4$, respectively, where μ_j is the j-th central moment and σ is the standard deviation. Since for some parameter values the moments of the MF distribution can not exist, alternative measures for the skewness and kurtosis based on quantiles, are sometimes more appropriate. The skewness measure S of Bowley and the kurtosis measure K of Moors are, respectively, defined by



(a) (b)

Figure 3.3: Plots of the MF pdf (3.7) for selected parameters.



(a) (b)

Figure 3.4: Plots of the MF hrf (3.8) for selected parameters.

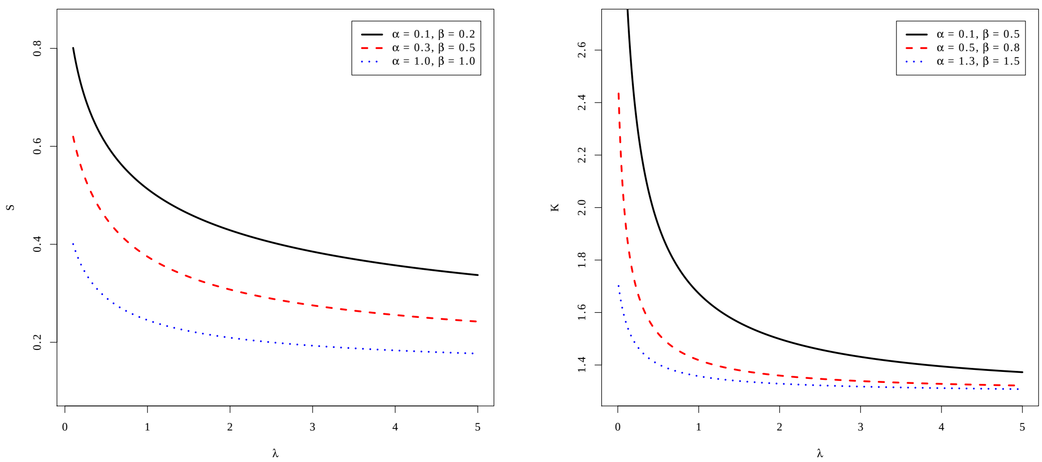
$$S = \frac{Q(6/8) + Q(2/8) - 2\,Q(4/8)}{Q(6/8) - Q(2/8)} \tag{3.14}$$

and

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}. \tag{3.15}$$

These measures are more robust and they exist even for distributions without moments.

Figure 3.5 displays plots of the skewness (4.18) and kurtosis (4.19) as functions of λ for some values of α and β. For evaluating the quantiles using the Lambert function given by (3.10), we use the R software (version 3.0.2, lamW package). These plots reveal that, in general, the skewness and kurtosis measures are decreasing functions of λ. This fact can be verified in Figure 3.3(a) for the skewness and in Figure 3.3(b) for the kurtosis.



(a) Skewness (b) Kurtosis

Figure 3.5: Plots of the skewness and kurtosis of the MF distribution for selected parameters.

3.6 Moments

Moments are important in any statistical analysis. Some of the most important features of a distribution can be studied through moments. For instance, the first four moments can be used to describe some characteristics of a distribution. For $r \in \mathbb{N}$, the r-th ordinary moment μ'_r of X is given by

$$\mu'_r = \mathbb{E}(X^r) = \int_0^\infty x^r f(x)\,dx = \int_0^\infty x^{r-1}\,(\beta+\lambda x)\left(\frac{\alpha}{x}\right)^\beta \exp\left[-\lambda x - \left(\frac{\alpha}{x}\right)^\beta \mathrm{e}^{-\lambda x}\right] dx. \tag{3.16}$$

For practical purposes, μ'_r can be evaluated numerically.

Also, for $z \geq 0$, the r-th incomplete moment of X is defined by

$$m_X^{(r)}(z) = \int_0^z x^r\, f(x)\,dx = \int_0^z x^{r-1}\,(\beta+\lambda x)\,\left(\frac{\alpha}{x}\right)^\beta \exp\left[-\lambda x - \left(\frac{\alpha}{x}\right)^\beta \mathrm{e}^{-\lambda x}\right] dx. \tag{3.17}$$

Clearly, $m_X^{(r)}(0) = 0$ and $\mu'_r = \lim_{z\to\infty} m_X^{(r)}(z)$. The following result gives an expansion for the r-th incomplete moment of X in terms of generalized exponential integrals.

PROPOSITION 4 *For $\lambda > 0$ and $0 < z \leq \frac{\beta}{\lambda} W_0(\mathrm{e}^{-1})$, the r-th incomplete moment of X can be expressed as*

$$m_X^{(r)}(z) = \sum_{n=r}^\infty c_n(z)\,\mathcal{E}_{n/\beta}\left[\left(\frac{\alpha}{z}\right)^\beta \mathrm{e}^{-\lambda z}\right],$$

where $c_n(z) = a_n\left[\left(\frac{\alpha}{z}\right)^\beta \mathrm{e}^{-\lambda z}\right]^{1-\frac{n}{\beta}}$, $a_n = \frac{r\,(-n)^{n-r}\alpha^n\lambda^{n-r}}{n\,(n-r)!\,\beta^{n-r}}$ and $\mathcal{E}_\nu(x) = \int_1^\infty \mathrm{e}^{-tx}\,t^{-\nu}\,dt$ is the generalized exponential-integral function.

Proof: Letting $t = \left(\frac{\alpha}{x}\right)^\beta \mathrm{e}^{-\lambda x}$ and after some algebraic manipulation, we have

$$\frac{\lambda\,x}{\beta}\,\mathrm{e}^{\lambda x/\beta} = \frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}.$$

Applying $W_0(\cdot)$ in both sides and using (3.1) gives

$$x = \frac{\beta}{\lambda}\,W_0\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right).$$

Using the result $\frac{dt}{dx} = -\frac{1}{x}(\beta+\lambda x)\left(\frac{\alpha}{x}\right)^\beta \mathrm{e}^{-\lambda x}$ in (3.17), we obtain

$$m_X^{(r)}(z) = \left(\frac{\beta}{\lambda}\right)^r \int_{(\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z}}^\infty W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right)\,\mathrm{e}^{-t}\,dt.$$

Using (3.3) and assuming that $\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}} \leq \mathrm{e}^{-1}$ gives

$$m_X^{(r)}(z) = \int_{(\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z}}^{\infty} \sum_{n=r}^{\infty} a_n \frac{\mathrm{e}^{-t}}{t^{n/\beta}}\, dt, \tag{3.18}$$

where

$$a_n = \frac{r\,(-n)^{n-r} \alpha^n \lambda^{n-r}}{n\,(n-r)!\,\beta^{n-r}}. \tag{3.19}$$

Since, for $r \in \mathbb{N}$, exists $M > 1$ such that $W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right) < t^r$ in (M,∞) and $t^r \mathrm{e}^{-t}$ is integrable in $(0,\infty)$, then $W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right) \mathrm{e}^{-t}$ is integrable in (M,∞). Therefore, by the continuity of $W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right) \mathrm{e}^{-t}$ in $[\varepsilon,M]$ for all $0 < \varepsilon < M$, we conclude that this function is integrable in $\left[(\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z},\infty\right)$, since $(\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z} > 0$. In addition, the functions $\frac{\mathrm{e}^{-t}}{t^{n/\beta}}$, $n = r, r+1, \ldots$, are integrable in $\left[(\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z},\infty\right)$. Thus, it is possible to exchange in (3.18) the infinite sum and the integral using the dominated convergence theorem. We obtain

$$m_X^{(r)}(z) = \sum_{n=r}^{\infty} a_n \int_{(\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z}}^{\infty} \frac{\mathrm{e}^{-t}}{t^{n/\beta}}\, dt. \tag{3.20}$$

The generalized exponential-integral function is defined by (CHICCOLI *et al.*, 1990)

$$\mathcal{E}_\nu(x) = \int_1^\infty \mathrm{e}^{-tx}\, t^{-\nu}\, dt, \quad x > 0, \nu \in \mathbb{R},$$

which is equivalent to

$$\mathcal{E}_\nu(x) = x^{\nu-1} \int_x^\infty \mathrm{e}^{-t}\, t^{-\nu}\, dt. \tag{3.21}$$

Thus, letting $\nu = \frac{n}{\beta}$ and $x = (\frac{\alpha}{z})^\beta \mathrm{e}^{-\lambda z}$ in (3.21) and replacing in (3.20), we obtain

$$m_X^{(r)}(z) = \sum_{n=r}^{\infty} c_n(z)\, \mathcal{E}_{n/\beta}\left[\left(\frac{\alpha}{z}\right)^\beta \mathrm{e}^{-\lambda z}\right], \tag{3.22}$$

where

$$c_n(z) = a_n \left[\left(\frac{\alpha}{z}\right)^\beta \mathrm{e}^{-\lambda z}\right]^{1-\frac{n}{\beta}}.$$

We have that the radius of converge in (3.3) is e^{-1} and therefore we must have that

$\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}} \leq \mathrm{e}^{-1}$, which is equivalent to $t \geq \left(\frac{\alpha\,\lambda}{\beta}\right)^{\beta} \mathrm{e}^{\beta}$. Since the lower limit $\left(\frac{\alpha\,\lambda}{\beta}\right)^{\beta} \mathrm{e}^{\beta}$ is attained in (3.18) for $z = \frac{\beta}{\lambda}\, W_0(\mathrm{e}^{-1})$ and the function $(\frac{\alpha}{z})^{\beta}\mathrm{e}^{-\lambda z}$ is decreasing, expansion (3.3) can be applied only for $0 < z \leq \frac{\beta}{\lambda}\, W_0(\mathrm{e}^{-1})$, which implies that (3.22) is only valid in this range. ■

From the above proof, we obtain the following result.

COROLLARY 2 *If $\beta > r$, then $\mu_r' < \infty$.*

Proof: If $\lambda = 0$, we obtain the Fréchet distribution, which is a well-known result. If $\lambda > 0$, we have that exists $M' > 0$ such that $W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right) < t^{-r/\beta}$ in $(0,M')$ for $\beta > r$ and since $t^{-r/\beta}\mathrm{e}^{-t}$ is integrable in $(0,\infty)$ for $\beta > r$, then $W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right)\mathrm{e}^{-t}$ also is integrable in $(0,M')$ for $\beta > r$. From the proof of the Proposition 4, we conclude that $W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right)\mathrm{e}^{-t}$ is integrable in $(0,\infty)$ for $\beta > r$ and therefore $\mu_r' < \infty$. ■

The following result provides an expansion for μ_r' by considering a suitable transformation in (3.16).

PROPOSITION 5 *For $\lambda > 0$ and $\mu_r' < \infty$, an expression for μ_r' is given by*

$$\mu_r' = \sum_{k=1}^{\infty}\sum_{n=r}^{\infty} p_1(n)\, I_n(k) + \sum_{n=r}^{\infty} p_2(n)\, \mathcal{E}_{n/\beta}\left[\left(\frac{\mathrm{e}\,\alpha\lambda}{\beta}\right)^{\beta}\right], \tag{3.23}$$

where

$$p_1(n) = \frac{2^{\beta}\alpha^{n}}{\beta^2}\left(\frac{\lambda}{\beta}\right)^{n-r}\frac{r(-n)^{n-r}}{n(n-r)!}, \quad p_2(n) = \frac{r(-n)^{n-r}}{n(n-r)!}\left(\frac{\beta}{\lambda}\right)^{r}\left(\frac{\mathrm{e}\,\alpha\lambda}{\beta}\right)^{\beta-n}$$

and

$$I_n(k) = (\vartheta_{k+1})^{1-\frac{n}{\beta}}\,\mathcal{E}_{n/\beta}(\vartheta_{k+1}) - (\vartheta_k)^{1-\frac{n}{\beta}}\,\mathcal{E}_{n/\beta}(\vartheta_k), \quad \vartheta_k = \left(\frac{\mathrm{e}\alpha\lambda}{2\beta k^{\beta}}\right)^{\beta}\left(\frac{\mathrm{e}\alpha\lambda}{\beta}\right)^{(1-\beta)\beta}.$$

Proof: Following the proof in Proposition 4, we can write

$$\mu_r' = \left(\frac{\beta}{\lambda}\right)^{r}\int_0^{\infty} W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right)\mathrm{e}^{-t}\,dt.$$

Setting $t_1 = (\frac{\mathrm{e}\,\alpha\lambda}{\beta})^{\beta}$ and dividing the integration interval, we can write $\mu_r' = \left(\frac{\beta}{\lambda}\right)^{r}(I_1 + I_2)$, where

$$I_1 = \int_0^{t_1} W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right)\mathrm{e}^{-t}\,dt, \quad I_2 = \int_{t_1}^{\infty} W_0^r\left(\frac{\alpha\,\lambda}{\beta\,t^{1/\beta}}\right)\mathrm{e}^{-t}\,dt.$$

Since $0 < \frac{\alpha\,\lambda}{\beta\, t^{1/\beta}} < \mathrm{e}^{-1}$ for $t_1 < t < \infty$, applying the expansion (3.3), we obtain

$$I_2 = \sum_{n=r}^{\infty} \frac{r(-n)^{n-r}}{n(n-r)!}\, t_1^{1-\frac{n}{\beta}}\, \mathcal{E}_{n/\beta}(t_1). \tag{3.24}$$

Next, we aim to find an expression for I_1. Note that, since the set $\left\{ t_k = (\frac{\mathrm{e}\,\alpha\lambda}{k\beta})^\beta \,:\, k \in \mathbb{N} \right\}$ is enumerable, it can be neglected of the integration interval of I_1. Further, since $\mu_r' < \infty$, the function $w(t) = W_0^r\left(\frac{\alpha\,\lambda}{\beta\, t^{1/\beta}}\right)\mathrm{e}^{-t}$ is integrable in $(0,t_1)$. Let $w_k(t) = w(t)\,\mathcal{I}_{(t_{k+1},t_k)}(t)$, where $\mathcal{I}_{(t_{k+1},t_k)}(t)$ is the indicator function of the interval (t_{k+1},t_k). Thus, we have $w(t) = \sum_{k=1}^{\infty} w_k(t)$ almost everywhere $t \in (0,t_1)$. Applying the dominated convergence theorem gives

$$I_1 = \sum_{k=1}^{\infty} \int_0^{t_1} w_k(t)\, dt. \tag{3.25}$$

Setting $y = \tan\left(\frac{\mathrm{e}\pi\alpha\lambda}{2\beta\, t^{1/\beta}}\right)$ for $t \in (t_{k+1},t_k)$ gives

$$\int_0^{t_1} w_k(t)\, dt = \frac{1}{\beta}\left(\frac{\mathrm{e}\pi\alpha\lambda}{\beta}\right)^\beta \int_{y_k}^{y_{k+1}} W_0^r\left(\frac{2}{\mathrm{e}\pi}\arctan(y)\right)\exp\left[-\left(\frac{\mathrm{e}\pi\alpha\lambda}{2\,\beta\arctan(y)}\right)^\beta\right] \frac{(1+y^2)^{-1}}{[\arctan(y)]^{\beta+1}}\, dy,$$

where $y_k = \tan\left[\pi k^\beta \left(\frac{\beta}{\mathrm{e}\,\alpha\,\lambda}\right)^{\beta-1}\right]$. Note that, for $t \in (t_{k+1},t_k)$, we have $\frac{\mathrm{e}\pi\alpha\lambda}{2\beta t^{1/\beta}} \neq \frac{\pi}{2}m$ $(m \in \mathbb{N})$ and, therefore, the above transformation is well-defined.

Since $|\arctan(y)| < \pi/2$, then $\left|\frac{2}{\mathrm{e}\pi}\arctan(y)\right| < \mathrm{e}^{-1}$ and, using the expansion (3.3), we obtain

$$\int_0^{t_1} w_k(t)\, dt = \sum_{n=r}^{\infty} \frac{r(-n)^{n-r}}{n(n-r)!}\, \frac{2^n}{\beta}\left(\frac{\alpha\lambda}{\beta}\right)^\beta (\mathrm{e}\pi)^{\beta-n} \int_{y_k}^{y_{k+1}} \exp\left[-\left(\frac{\mathrm{e}\pi\alpha\lambda}{2\,\beta\arctan(y)}\right)^\beta\right] \frac{[\arctan(y)]^{n-\beta-1}}{(1+y^2)}\, dy.$$

Setting $z = \arctan(y)$, we have

$$\int_0^{t_1} w_k(t)\, dt = \sum_{n=r}^{\infty} \frac{r(-n)^{n-r}}{n(n-r)!}\, \frac{2^n}{\beta}\left(\frac{\alpha\lambda}{\beta}\right)^\beta (\mathrm{e}\pi)^{\beta-n} \int_{z_k}^{z_{k+1}} \exp\left[-\left(\frac{\mathrm{e}\pi\alpha\lambda}{2\beta z}\right)^\beta\right] z^{n-\beta-1}\, dz, \tag{3.26}$$

where $z_k = \pi k^\beta \left(\frac{\beta}{\mathrm{e}\,\alpha\,\lambda}\right)^{\beta-1}$. Setting $\vartheta = \left(\frac{\mathrm{e}\pi\alpha\lambda}{2\beta z}\right)^\beta$, we obtain

$$\int_{z_k}^{z_{k+1}} \exp\left[-\left(\frac{\mathrm{e}\pi\alpha\lambda}{2\beta z}\right)^\beta\right] z^{n-\beta-1}\, dz = \frac{1}{\beta}\left(\frac{\mathrm{e}\pi\alpha\lambda}{2\beta}\right)^{n-\beta} \int_{\vartheta_{k+1}}^{\vartheta_k} \mathrm{e}^{-\vartheta}\, \vartheta^{-n/\beta}\, d\vartheta, \tag{3.27}$$

where $\vartheta_k = \left(\frac{e\pi\alpha\lambda}{2\beta\, z_k}\right)^{\beta}$.

Replacing (3.27) in equation (3.26) and considering the expression (3.25) gives

$$I_1 = \sum_{k=1}^{\infty}\sum_{n=r}^{\infty}\frac{r(-n)^{n-r}}{n(n-r)!}\,\frac{2^{\beta}}{\beta^2}\left(\frac{\alpha\lambda}{\beta}\right)^{n}\left[(\vartheta_{k+1})^{1-\frac{n}{\beta}}\,\mathcal{E}_{n/\beta}(\vartheta_{k+1}) - (\vartheta_k)^{1-\frac{n}{\beta}}\,\mathcal{E}_{n/\beta}(\vartheta_k)\right], \tag{3.28}$$

where $\mathcal{E}_{\nu}(x)$ is defined in (3.21).

Finally, we obtain from equations (3.24) and (3.28),

$$\mu'_r = \sum_{k=1}^{\infty}\sum_{n=r}^{\infty} p_1(n)\, I_n(k) + \sum_{n=r}^{\infty} p_2(n)\,\mathcal{E}_{n/\beta}(t_1),$$

where

$$p_1(n) = \frac{2^{\beta}\alpha^n}{\beta^2}\left(\frac{\lambda}{\beta}\right)^{n-r}\frac{r(-n)^{n-r}}{n(n-r)!}, \quad p_2(n) = \frac{r(-n)^{n-r}}{n(n-r)!}\left(\frac{\beta}{\lambda}\right)^{r} t_1^{1-\frac{n}{\beta}}$$

and

$$I_n(k) = (\vartheta_{k+1})^{1-\frac{n}{\beta}}\,\mathcal{E}_{n/\beta}(\vartheta_{k+1}) - (\vartheta_k)^{1-\frac{n}{\beta}}\,\mathcal{E}_{n/\beta}(\vartheta_k).$$

■

3.6.1 Generating function

The moment generating function (mgf) of X, say $M(s) = \mathbb{E}(e^{sX})$, $s \geq 0$, is given by

$$M(s) = \int_0^{\infty}\frac{1}{x}\,(\beta + \lambda\, x)\,\left(\frac{\alpha}{x}\right)^{\beta}\exp\left[(s-\lambda)\,x - \left(\frac{\alpha}{x}\right)^{\beta} e^{-\lambda x}\right]\, dx.$$

Also, for $z \geq 0$, we define the incomplete mgf of X by

$$M_z(s) = \int_0^{z}\frac{1}{x}\,(\beta + \lambda\, x)\,\left(\frac{\alpha}{x}\right)^{\beta}\exp\left[(s-\lambda)\,x - \left(\frac{\alpha}{x}\right)^{\beta} e^{-\lambda x}\right]\, dx.$$

For $r \in \mathbb{N}$, we have $M^{(r)}(0) = \frac{d^r}{dx^r} M(s)\,\big|_{s=0} = \mu'_r$ and $M_z^{(r)}(0) = \frac{d^r}{dx^r} M_z(s)\,\big|_{s=0} = m_X^{(r)}(z)$.

Using the expansion (3.4) we obtain the following result.

PROPOSITION 6 *For $\lambda > 0$ and $0 < z \leq \frac{\beta}{\lambda} W_0(e^{-1})$, the incomplete mgf of X can be* *expanded as*

$$M_z(s) = \sum_{n=0}^{\infty} d_n(z,s)\, \mathcal{E}_{n/\beta}\left[\left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}\right],$$

where $d_n(z,s) = b_n(s)\left[\left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}\right]^{1-\frac{n}{\beta}}$ *and* $b_n(s) = \frac{s\left(s-\frac{\lambda n}{\beta}\right)^{n-1}\alpha^n}{n!}$.

Proof: Letting $t = \left(\frac{\alpha}{x}\right)^{\beta} \mathrm{e}^{-\lambda x}$ and following the proof of the Proposition 4, we obtain

$$M_z(s) = \int_{\left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}}^{\infty} \exp\left[\frac{\beta s}{\lambda} W_0\left(\frac{\alpha\lambda}{\beta t^{1/\beta}}\right) - t\right] dt.$$

Then, for $0 < z \leq \frac{\beta}{\lambda} W_0(\mathrm{e}^{-1})$, we have from (3.4)

$$M_z(s) = \sum_{n=0}^{\infty} b_n(s) \int_{\left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}}^{\infty} \frac{\mathrm{e}^{-t}}{t^{n/\beta}}\, dt, \tag{3.29}$$

where

$$b_n(s) = \frac{s\left(s - \frac{\lambda n}{\beta}\right)^{n-1} \alpha^n}{n!}.$$

Letting $\nu = \frac{n}{\beta}$ and $x = \left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}$ in equation (3.21) and replacing in (3.29), we obtain

$$M_z(s) = \sum_{n=0}^{\infty} d_n(z,s)\, \mathcal{E}_{n/\beta}\left[\left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}\right], \tag{3.30}$$

where

$$d_n(z,s) = b_n(s)\left[\left(\frac{\alpha}{z}\right)^{\beta} \mathrm{e}^{-\lambda z}\right]^{1-\frac{n}{\beta}}.$$

■

We can obtain an expansion for $M(s)$ using expansion (3.4) and following the insights of the proof in Proposition 5. The result is given in the following proposition.

PROPOSITION 7 *For* $\lambda > 0$*, an expression for* $M(s)$*,* $s \geq 0$*, is given by*

$$M(s) = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} p_1(n,s)\, I_n(k) + \sum_{n=0}^{\infty} p_2(n,s)\, \mathcal{E}_{n/\beta}\left[\left(\frac{\mathrm{e}\alpha\lambda}{\beta}\right)^{\beta}\right], \tag{3.31}$$

where

$$p_1(n,s) = \frac{(2\alpha)^\beta s(\beta s - n\lambda)^{n-1}}{\beta^{n+2}\, n!}, \quad p_2(n,s) = \frac{s(\beta s - n\lambda)^{n-1}\alpha^n}{\beta^{n-1}\, n!}\left(\frac{\mathrm{e}\alpha\lambda}{\beta}\right)^{\beta-n}$$

and $I_n(k)$ is given in Proposition 5.

Proof: From the proof in Proposition 4, we have $M(s) = \int_0^\infty \nu(t)\, dt$, where

$$\nu(t) = \exp\left[\frac{\beta\, s}{\lambda}\, W_0\!\left(\frac{\alpha\,\lambda}{\beta\, t^{1/\beta}}\right) - t\right].$$

Setting $t_k = (\frac{\mathrm{e}\,\alpha\lambda}{k\beta})^\beta$, $k \in \mathbb{N}$, and following the proof in Proposition 5, we have $M(s) = I_1 + I_2$, where

$$I_1 = \sum_{k=1}^{\infty} \int_0^{t_1} \nu_k(t)\, dt, \quad I_2 = \int_{t_1}^{\infty} \nu(t)\, dt$$

and $\nu_k(t) = \nu(t)\, \mathcal{I}_{(t_{k+1},t_k)}(t)$.

From the expansion (3.4), we obtain

$$I_2 = \sum_{n=0}^{\infty} p_2(n,s)\, \mathcal{E}_{n/\beta}(t_1),$$

where $p_2(n,s) = \frac{s(\beta s - n\lambda)^{n-1}\alpha^n}{\beta^{n-1}\, n!}\, t_1^{1-n/\beta}$.

Following the same sequence of transformations as in the proof of Proposition 5, we can write

$$\int_0^{t_1} \nu_k(t)\, dt = \sum_{n=0}^{\infty} \frac{(2\alpha)^\beta s(\beta s - n\lambda)^{n-1}}{\beta^{n+2}\, n!} \int_{\vartheta_{k+1}}^{\vartheta_k} \mathrm{e}^{-\vartheta}\, \vartheta^{-n/\beta}\, d\vartheta,$$

where $\vartheta_k = \left(\frac{\mathrm{e}\alpha\lambda}{2\beta k^\beta}\right)^\beta \left(\frac{\mathrm{e}\alpha\lambda}{\beta}\right)^{(1-\beta)\beta}$. Thus, we have

$$I_1 = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{(2\alpha)^\beta s(\beta s - n\lambda)^{n-1}}{\beta^{n+2}\, n!}\, I_n(k),$$

where $I_n(k)$ is given in Proposition 5.

Finally, we have

$$M(s) = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} p_1(n,s)\, I_n(k) + \sum_{n=0}^{\infty} p_2(n,s)\, \mathcal{E}_{n/\beta}(t_1),$$

where $p_1(n,s) = \frac{(2\alpha)^\beta s(\beta s - n\lambda)^{n-1}}{\beta^{n+2}\, n!}$. ■

Equations (3.22), (3.23), (3.30) and (3.31) are the main results of this section.

3.7 Mean deviations and Bonferroni and Lorenz curves

The mean deviations of X about the mean, $\delta_1 = \mathbb{E}|X - \mu_1'|$, and about the median, $\delta_2 = \mathbb{E}|X - M|$, used frequently as measures of dispersion, can be expressed as

$$\delta_1 = 2\,\mu_1'\,F(\mu_1') - 2\,m_X^{(1)}(\mu_1') \text{ and } \delta_2 = \mu_1' - 2\,m_X^{(1)}(M),$$

where $M = Q(0.5)$ is the median of X obtained from (3.10) at $u = 0.5$ and the quantities μ_1' and $m_X^{(1)}(z)$ can be evaluated numerically from (3.16) and (3.17), respectively.

The first incomplete moment can be applied to obtain the Bonferroni and Lorenz curves, which are useful in several fields. The Bonferroni and Lorenz curves are defined, respectively, by

$$B(\pi) = \frac{m_X^{(1)}(q)}{\pi\,\mu_1'} \text{ and } L(\pi) = \pi\,B(\pi),$$

where $q = Q(\pi)$ is evaluated from (3.10) for $0 < \pi < 1$. From the Proposition 4, we obtain the following expansion for the Bonferroni curve.

COROLLARY 3 *For $\lambda > 0$ and $0 < \pi \leq \mathrm{e}^{-\left(\frac{\alpha\lambda}{\beta}\right)^\beta \mathrm{e}^\beta}$, the Bonferroni curve can be expanded as*

$$B(\pi) = \frac{1}{\pi\,\mu_1'} \sum_{n=1}^{\infty} a_n \left(-\log\pi\right)^{1-\frac{n}{\beta}} \mathcal{E}_{n/\beta}(-\log\pi)\,,$$

where a_n is given by (3.19) for $r = 1$.

Proof:

From equation (3.10), we have

$$q = Q(\pi) = \frac{\beta}{\lambda}\,W_0\left(\frac{\alpha\,\lambda}{\beta\left[\log\left(\frac{1}{\pi}\right)\right]^{1/\beta}}\right).$$

Thus, after some algebraic manipulation and using (3.2), we have

$$\left(\frac{\alpha}{q}\right)^{\beta} \mathrm{e}^{-\lambda q} = \left(\frac{\beta\, W_0\left\{\frac{\alpha\lambda}{\beta\left[\log\left(\frac{1}{\pi}\right)\right]^{1/\beta}}\right\}}{\alpha\,\lambda}\right)^{-\beta} \exp\left(-\beta\, W_0\left\{\frac{\alpha\lambda}{\beta\left[\log\left(\frac{1}{\pi}\right)\right]^{1/\beta}}\right\}\right) = -\log\pi.$$

Finally, from equation (3.22), we obtain

$$B(\pi) = \frac{1}{\pi\,\mu_1'}\, m_X^{(1)}(q) = \frac{1}{\pi\,\mu_1'} \sum_{n=1}^{\infty} a_n\,(-\log\pi)^{1-\frac{n}{\beta}}\, \mathcal{E}_{n/\beta}(-\log\pi)\,, \tag{3.32}$$

where a_n is given by (3.19) for $r = 1$. This expansion holds only for $0 < q \leq \frac{\beta}{\lambda}\, W_0(\mathrm{e}^{-1})$, which is equivalent to $0 < \pi \leq \mathrm{e}^{-\left(\frac{\alpha\lambda}{\beta}\right)^{\beta}\mathrm{e}^{\beta}}$. ■

Equation (3.32) is the main result of this section.

3.8 Order statistics

Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ be the order statistics of a random sample of size n from the distribution $F(x)$. Then, for $m = 1,2,\ldots,n$, the pdf of the m-th order statistic, $X_{(m)}$, is given by (SEVERINI, 2005, p. 218)

$$f_{(m)}(x) = K\, F^{m-1}(x)\,[1 - F(x)]^{n-m}\, f(x),$$

where $K = n!/[(m-1)!\,(n-m)!]$. The cdf of $X_{(m)}$ is given by

$$F_{(m)}(x) = \sum_{j=m}^{n} \binom{n}{j} F^{j}(x)\,[1 - F(x)]^{n-j}.$$

In particular, the cdf's of $X_{(n)}$ and $X_{(1)}$ are given, respectively, by

$$F_{(n)}(x) = F^{n}(x), \quad F_{(1)}(x) = 1 - [1 - F(x)]^{n}. \tag{3.33}$$

Let $Q_{(m)}(u)$ be (for $0 < u < 1$) the qf of $X_{(m)}$. Then, we obtain from (3.33)

$$Q_{(n)}(u) = Q(u^{1/n}), \quad Q_{(1)}(u) = Q[1 - (1-u)^{1/n}]. \tag{3.34}$$

where $Q(\cdot)$ is the qf of X. Thus, from (3.10) and (3.34), we can write the qf's of $X_{(n)}$ and $X_{(1)}$ in closed-form in terms of the Lambert W function.

It is possible, when $\mu_r' < \infty$, obtain an expression for the r-th ordinary moment of the order statistics using a result given by Barakat & Abdelkader (2004) for the case of i.i.d. random variables. Thus, if $\mu'_r < \infty$, we can express the r-th moment of the m-th order statistic $X_{(m)}$ as (SILVA *et al.*, 2010)

$$\mu_{(m)}^{(r)} = \mathbb{E}(X_{(m)}^r) = \sum_{j=n-m+1}^{n} (-1)^{j-n+m-1} \binom{j-1}{n-m} \binom{n}{j} I_j(r),$$

where $I_j(r) = r\int_0^\infty x^{r-1}[1-F(x)]^j\,dx$. In particular, for the MF distribution, we obtain

$$\mu_{(m)}^{(r)} = r \sum_{j=n-m+1}^{n} (-1)^{j-n+m-1} \binom{j-1}{n-m} \binom{n}{j} \int_0^\infty x^{r-1} \left[1 - \mathrm{e}^{-\left(\frac{\alpha}{x}\right)^\beta \mathrm{e}^{-\lambda x}}\right]^j dx,$$

where the last integral can be evaluated numerically.

3.9 Maximum Likelihood Estimation

Several approaches for parameter estimation were proposed in the statistical literature but the maximum likelihood method is the most commonly employed. The MLE's enjoy desirable properties for constructing confidence intervals. In this section, we consider the estimation of the parameters of the MF distribution by this method. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$ be a sample of size n from $X \sim \mathrm{MF}(\alpha,\beta,\lambda)$ and let $\boldsymbol{\theta} = (\alpha,\beta,\lambda)^\top$ be the parameter vector. The log-likelihood for the sample $\boldsymbol{x}$, denoted by $\ell_f(\boldsymbol{\theta};\boldsymbol{x})$, is given by

$$\begin{aligned}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = {} & n\,\beta\log(\alpha) - (\beta+1)\sum_{i=1}^n \log(x_i) + \sum_{i=1}^n \log(\beta + \lambda\, x_i) \\ & - \lambda\sum_{i=1}^n x_i - \alpha^\beta \sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta}. \end{aligned} \tag{3.35}$$

The MLE $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ can be obtained by maximizing (3.35) directly by using a scientific library. Alternatively, we can obtain the components of the score vector $\boldsymbol{U}_{\boldsymbol{\theta}} = (U_\alpha,U_\beta,U_\lambda)^\top$ and set them to zero. They are given by

$$U_\alpha = \frac{\partial}{\partial\alpha}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = \frac{n\,\beta}{\alpha} - \beta\,\alpha^{\beta-1}\sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta}, \tag{3.36}$$

$$U_\beta = \frac{\partial}{\partial\beta}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = n\log(\alpha) - \sum_{i=1}^n \log(x_i) + \sum_{i=1}^n \frac{1}{\beta+\lambda x_i} - \alpha^\beta \log(\alpha)\sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta} + \alpha^\beta \sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}\log(x_i)}{x_i^\beta},$$

and

$$U_\lambda = \frac{\partial}{\partial\lambda}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) = \sum_{i=1}^n \frac{x_i}{\beta+\lambda x_i} + \alpha^\beta \sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}}{x_i^{\beta-1}} - \sum_{i=1}^n x_i. \tag{3.37}$$

The MLE $\hat{\boldsymbol{\theta}}_n$ can be determined by setting $U_\alpha = U_\beta = U_\lambda = 0$ and by solving these equations simultaneously. Because they can not be solved in closed-form, numerical iterative methods, such as Newton-Raphson type algorithms, can be applied.

Under general regularity conditions, we have $(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \overset{a}{\sim} N_3(\mathbf{0}, K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the 3×3 expected information matrix and $\overset{a}{\sim}$ denotes the asymptotic distribution. For n large, $K(\boldsymbol{\theta})$ can be approximated by the observed information matrix. This normal approximation for the MLE $\hat{\boldsymbol{\theta}}_n$ can be used for constructing approximate confidence intervals and for testing hypotheses on the parameters α, β and λ.

In many cases, it is of interest to perform inference about some parameters of the model by assuming that the remaining parameters are known.

PROPOSITION 8 *Let $\boldsymbol{x} = (x_1,\ldots,x_n)^\top$ be a sample of size n from $X \sim \mathrm{MF}(\alpha,\beta,\lambda)$, with log-likelihood for $\boldsymbol{\theta}$ given by* (3.35). *Then*

a) If β and λ are known, then the MLE of α always exists and is unique, and is given by

$$\hat{\alpha}_n = \left(\frac{1}{n}\sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta}\right)^{-1/\beta}.$$

b) Assume α and β known and let $x_{(n)} = \max\{x_1,\ldots,x_n\}$. If $\alpha > x_{(n)}$, then the MLE of λ exists and is unique.

Proof: Suppose β and λ known. From (3.36) and considering that $\alpha, \beta > 0$ and $n > 0$, we have

$$U_\alpha = 0 \iff \alpha = \left(\frac{1}{n}\sum_{i=1}^n \frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta}\right)^{-1/\beta}. \tag{3.38}$$

Differentiating U_α with respect to α, we obtain

$$U_{\alpha\alpha} = -\frac{n\beta}{\alpha^2} - \beta(\beta-1)\alpha^{\beta-2}\sum_{i=1}^{n}\frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta}.$$

To see that the critical point given in (3.38) is a point of maximum, it has to satisfy $U_{\alpha\alpha} < 0$. But

$$U_{\alpha\alpha} < 0 \Longleftrightarrow \alpha^{-\beta} > (1-\beta)\frac{1}{n}\sum_{i=1}^{n}\frac{\mathrm{e}^{-\lambda x_i}}{x_i^\beta}.$$

Since the critical point cleary satisfies this condition, we conclude that the MLE of α always exist and is unique, and is given by (3.38). It proves (a).

To prove (b), suppose α and β known. Differentiating U_λ with respect to λ, we obtain

$$U_{\lambda\lambda} = -\sum_{i=1}^{n}\left[\frac{x_i}{(\beta+\lambda x_i)^2} + \frac{\alpha^\beta \mathrm{e}^{-\lambda x_i}}{x_i^{\beta-2}}\right].$$

Since each term in the above sum is positive, then $U_{\lambda\lambda} < 0$. Further, from (3.37), we have

$$\lim_{\lambda\to\infty} U_\lambda = -\sum_{i=1}^{n} x_i < 0, \tag{3.39}$$

and

$$\lim_{\lambda\to 0} U_\lambda = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{\alpha^\beta - x_i^\beta}{x_i^{\beta-1}}. \tag{3.40}$$

Thus, if $\alpha > x_{(n)}$, then $\lim_{\lambda\to 0} U_\lambda > 0$, which ensures the existence and uniqueness of the MLE of λ. ■

Conditions for the existence of the MLE of β, given that α and λ are known, are much more difficult to obtain and, therefore, are omitted from the above result.

Suppose that the parameter vector is partitioned as $\boldsymbol{\theta} = (\boldsymbol{\psi}_1^\top, \boldsymbol{\psi}_2^\top)^\top$, where $\dim(\boldsymbol{\psi}_1) + \dim(\boldsymbol{\psi}_2) = \dim(\boldsymbol{\theta})$. The likelihood ratio (LR) statistic for testing the null hypothesis $\mathcal{H}_0 : \boldsymbol{\psi}_1 = \boldsymbol{\psi}_1^{(0)}$ against the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\psi}_1 \neq \boldsymbol{\psi}_1^{(0)}$ is given by $\mathrm{LR}_n = 2\{\ell_f(\hat{\boldsymbol{\theta}}_n) - \ell_f(\tilde{\boldsymbol{\theta}}_n)\}$, where $\hat{\boldsymbol{\theta}}_n = (\hat{\boldsymbol{\psi}}_1^\top, \hat{\boldsymbol{\psi}}_2^\top)^\top$, $\tilde{\boldsymbol{\theta}}_n = (\boldsymbol{\psi}_1^{(0)^\top}, \tilde{\boldsymbol{\psi}}_2^\top)^\top$, $\hat{\boldsymbol{\psi}}_i$ and $\tilde{\boldsymbol{\psi}}_i$ are the MLE's under the alternative and null hypotheses, respectively, and $\boldsymbol{\psi}_1^{(0)}$ is a specified parameter vector. Based on the first-order asymptotic theory, we know that $\mathrm{LR}_n \overset{a}{\sim} \chi_k^2$, where $k = \dim(\boldsymbol{\psi}_1)$. Thus, we can test submodels of the MF distribution and analyze how significant are the parameters tested for modeling a particular data set.

3.10 Simulation study

In this section, we perform a Monte Carlo simulation experiment in order to evaluate the behavior of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{\alpha}_n, \hat{\beta}_n, \hat{\lambda}_n)$ in finite samples and estimate the relative bias, mean squared error (MSE), skewness and kurtosis for the sample sizes $n = 100, 200$ and 300. We consider $10{,}000$ Monte Carlo replications and use the BFGS method with analytical derivatives for maximizing the log-likelihood function (3.35). All computations are performed using the `C` programming language and the `GNU` Scientific Library (version 2.1).

The results, given in Table 3.1, reveal that the relative bias and MSE values decrease when n increases, which is to be expected since the MLE's are asymptotically unbiased. In addition, the skewness value decrease to zero and the kurtosis value decrease to 3.0 when n increases, which is to be expected due to the asymptotic normality of the MLE's. The values in this table also reveal that the relative bias and MSE for $\hat{\lambda}_n$ increases as the value of β increases. We can also note that the relative bias and MSE do not exceed, in absolute value, 0.2 and 0.3, respectively. Further, it can be noted in Table 3.1 that the parameter β was underestimated in some cases (negative relative bias).

3.11 Application

In this section, the potentiality of the MF distribution is illustrated by means of one application. We use a data set corresponding to 202 observations of plasma ferritin concentration in athletes (WEISBERG, 2005, Sec. 6.4) and fit the Fréchet (Fr), exponentiated Fréchet (EF), Marshall-Olkin Fréchet (MOF) and MF distributions to these data. The data are also used by Alizadeh *et al.* (2015). All computations are performed using the `R` software (version 3.0.2, `AdequacyModel` package). The descriptive statistics for this data set are given in Table 3.2.

For maximizing the log-likelihood function (3.35), we use the BFGS method with numerical derivatives. Initial values α_0, β_0 and λ_0 for the BFGS method can be obtained, based on Proposition 8, by taking $\alpha_0 > x_{(n)}$ and giving arbitrary positive values for β_0 and λ_0. For purposes of comparison, we compute some goodness-of-fit statistics: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Cramér-von Mises Criterion (W*) and Anderson-Darling Criterion (A*) (CHEN; BALAKRISHNAN, 1995). In general, the smaller the values of these statistics are, the better the fit is.

Table 3.1: Relative bias, MSE, skewness and kurtosis values of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{\alpha}_n, \hat{\beta}_n, \hat{\lambda}_n)$ for the MF model

				relative bias			MSE			skewness			kurtosis		
α	β	λ	n	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\hat{\lambda}_n$	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\hat{\lambda}_n$	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\hat{\lambda}_n$	$\hat{\alpha}_n$	$\hat{\beta}_n$	$\hat{\lambda}_n$
0.5	0.5	0.25	100	0.075	0.013	0.044	0.027	0.004	0.005	1.165	0.344	0.701	8.682	3.247	3.836
			200	0.036	0.006	0.022	0.010	0.002	0.003	0.990	0.246	0.499	4.994	3.191	3.466
			300	0.025	0.004	0.015	0.007	0.001	0.002	0.706	0.162	0.392	3.876	3.047	3.135
		0.5	100	0.098	0.017	0.030	0.043	0.006	0.017	2.409	0.377	0.577	14.488	3.290	3.551
			200	0.046	0.008	0.015	0.015	0.003	0.008	1.365	0.270	0.418	6.949	3.212	3.336
			300	0.031	0.005	0.010	0.009	0.002	0.005	0.943	0.182	0.330	4.587	3.060	3.053
	1.0	0.25	100	0.054	−0.002	0.160	0.009	0.016	0.024	1.611	0.155	1.084	8.294	3.114	4.778
			200	0.026	−0.002	0.082	0.003	0.008	0.011	0.935	0.120	0.789	4.870	3.108	4.005
			300	0.018	−0.002	0.055	0.002	0.005	0.007	0.651	0.067	0.631	3.853	3.034	3.511
		0.5	100	0.076	0.001	0.094	0.019	0.023	0.060	2.553	0.158	0.830	16.232	3.099	4.025
			200	0.035	0.000	0.049	0.006	0.011	0.028	1.374	0.121	0.609	6.896	3.098	3.602
			300	0.024	−0.000	0.032	0.004	0.007	0.018	0.946	0.068	0.483	4.705	3.016	3.247
1.0	0.5	0.25	100	0.069	0.022	0.022	0.119	0.005	0.004	1.519	0.445	0.577	7.845	3.313	3.620
			200	0.042	0.009	0.014	0.056	0.003	0.002	1.108	0.289	0.406	5.189	3.194	3.318
			300	0.031	0.005	0.010	0.036	0.002	0.001	0.922	0.186	0.328	4.495	3.055	3.054
		0.5	100	0.108	0.029	0.013	0.255	0.008	0.014	3.289	0.478	0.482	29.444	3.374	3.418
			200	0.060	0.013	0.009	0.098	0.004	0.007	1.718	0.322	0.347	9.246	3.217	3.250
			300	0.042	0.008	0.006	0.059	0.002	0.005	1.263	0.217	0.275	5.818	3.068	3.005
	1.0	0.25	100	0.072	0.002	0.091	0.066	0.023	0.015	1.884	0.197	0.797	9.031	3.055	3.912
			200	0.035	0.000	0.048	0.025	0.011	0.007	1.320	0.125	0.603	6.360	3.092	3.589
			300	0.024	−0.000	0.032	0.015	0.007	0.005	0.947	0.068	0.484	4.704	3.017	3.247
		0.5	100	0.123	0.008	0.053	0.238	0.035	0.041	7.889	0.213	0.625	178.195	3.061	3.558
			200	0.056	0.003	0.029	0.059	0.018	0.020	2.223	0.133	0.476	13.680	3.086	3.371
			300	0.036	0.002	0.019	0.032	0.012	0.013	1.388	0.080	0.375	6.555	3.000	3.097

For completeness purposes, we also fit and include in the comparison the exponentiated Weibull (EW) distribution (MUDHOLKAR; SRIVASTAVA, 1993), since it is a widely used lifetime model. Its cdf and pdf are given, respectively, by

$$H(x) = \left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\lambda} \quad \text{and} \quad h(x) = \frac{\beta\lambda}{\alpha}\,\left(\frac{x}{\alpha}\right)^{\beta-1}\,\left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\lambda-1}\,\mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}},$$

where $x \geq 0$ and $\alpha,\beta,\lambda > 0$.

The MLE's are given in Table 3.3 with their standard errors in parentheses and the goodness-of-fit values for the fitted distributions are listed in Table 3.4.

Table 3.2: Descriptive statistics for the plasma ferritin data

min.	1st quantile	median	mean	3rd quantile	max.
8.00	41.25	65.50	76.88	97.00	234.00

Table 3.3: MLE's and standard errors for the plasma ferritin data

	MLE		
Distribution	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
Fr(α,β)	46.800 (2.301)	1.520 (0.073)	-
EF(α,β,λ)	304.983 (133.421)	0.652 (0.086)	10.783 (4.547)
MOF(α,β,λ)	11.224 (2.747)	2.758 (0.162)	122.932 (93.599)
MF(α,β,λ)	181.980 (71.163)	0.704 (0.116)	0.017 (0.002)
EW(α,β,λ)	24.970 (12.537)	0.812 (0.167)	5.840 (3.174)

Table 3.4: Goodness-of-fit statistics for the plasma ferritin data

	Statistic				
Distribution	AIC	BIC	HQIC	W*	A*
Fr($\hat{\alpha}$,$\hat{\beta}$)	2113.022	2119.638	2115.699	0.410	2.739
EF($\hat{\alpha}$,$\hat{\beta}$,$\hat{\lambda}$)	2066.946	2076.870	2070.961	0.035	0.274
MOF($\hat{\alpha}$,$\hat{\beta}$,$\hat{\lambda}$)	2069.222	2079.147	2073.238	0.064	0.408
MF($\hat{\alpha}$,$\hat{\beta}$,$\hat{\lambda}$)	2064.734	2074.659	2068.750	**0.025**	**0.222**
EW($\hat{\alpha}$,$\hat{\beta}$,$\hat{\lambda}$)	**2063.647**	**2073.572**	**2067.663**	0.032	0.244

Since the Fréchet distribution is a submodel of the MF distribution, a comparison between them can be conducted by considering the AIC, BIC and HQIC statistics. However, since the EW, EF, MOF and MF distributions are non-nested models, a comparison among them is more appropriate by considering the W* and A* statistics. The figures in Table 3.4 reveal that the MF distribution has the smallest values of the W* and A* statistics among the fitted models and the smaller values of all statistics comparatively with the Fréchet distribution. Therefore, we can conclude that the MF distribution gives the best fit to the current data. The plots in Figure 3.6 display the Fréchet, EW, MOF, EF and MF estimated densities. Based on these plots, it is possible to assess the overall best fit of the MF and EW distributions.

A graphical analysis of the quality of fit can be assessed by means of Q-Q plots. So, in Figure 3.7 are given Q-Q plots for all current distributions. From these plots, we can observe that the MF distribution gives the best fit to the current data among the Fréchet, EF and MOF distributions, and a fit similar to that given by the EW distribution. Figure 3.7(a) reveal a less precise fit of the MF distribution in the tail, which can be explained by a more heavier tail comparatively with the Fréchet distribution (see Figure 3.6). This graphical analysis is consistent with the quantitative analysis given by the goodness-of-fit statistics listed in Table 3.4.

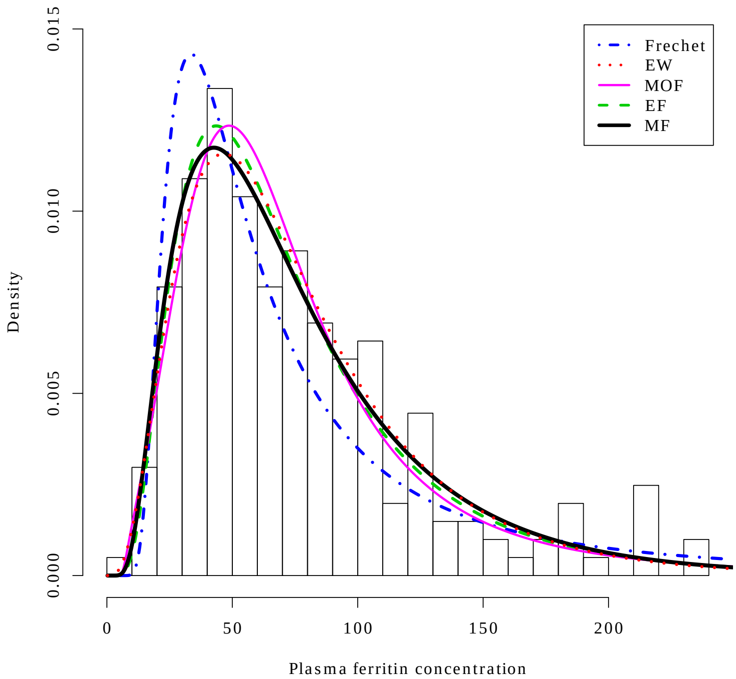


Figure 3.6: The Fréchet, EW, MOF, EF and MF estimated densities for the plasma ferritin data.

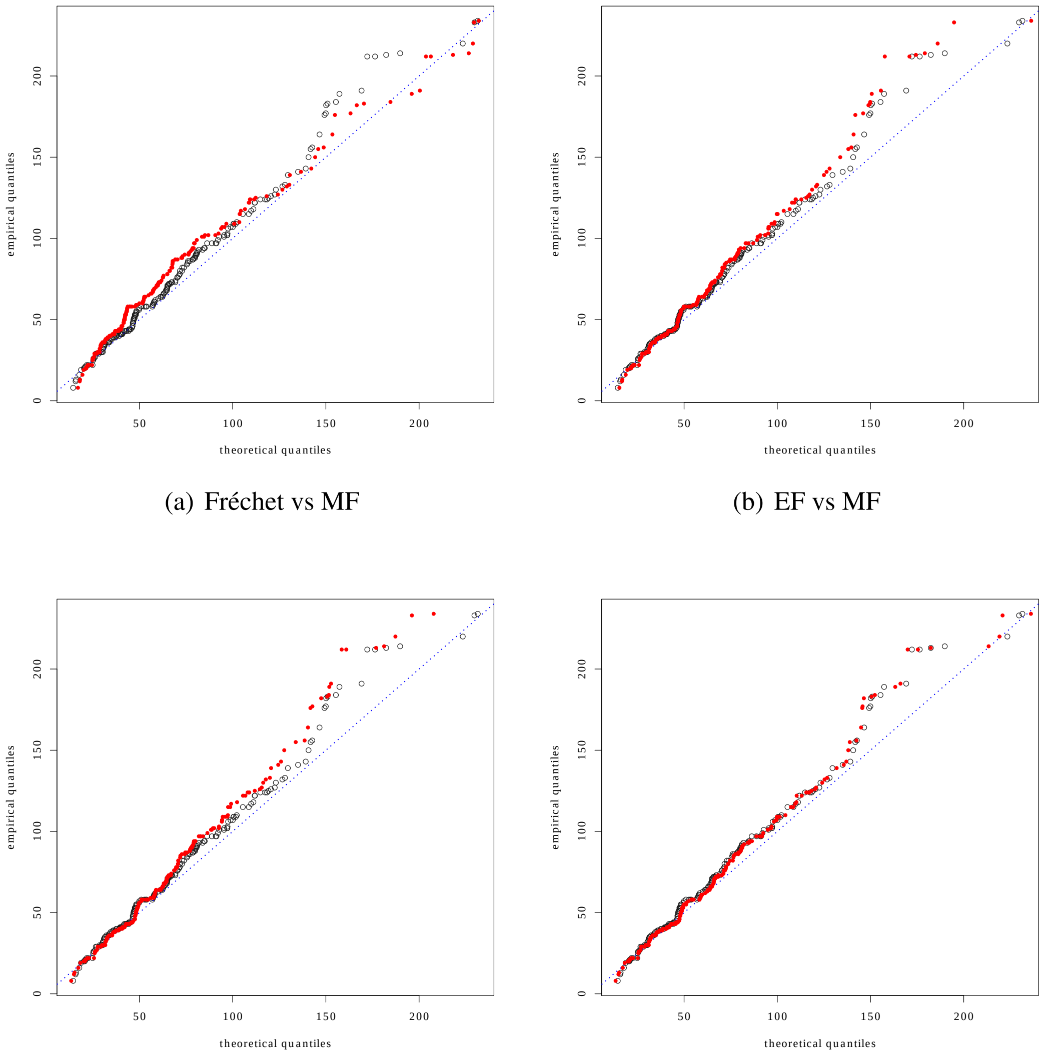
To analyze how significant is the additional parameter λ of the MF distribution for modeling the current data, we use the LR statistic, as discussed in Section 3.9, for testing the Fréchet model versus the MF model. The results are given in Table 3.5. We note that the rejection of the null hypothesis is significant. So, we have evidence of the potential need for the inclusion of the parameter λ in the Fréchet distribution for modeling the current data.

Table 3.5: LR test for the plasma ferritin data

Models	Hypoteses	LR statistic	p-value
Fréchet vs MF	$\mathcal{H}_0$: $\lambda = 0$ vs $\mathcal{H}_1$: $\lambda > 0$	50.288	1.33×10^{-12}

3.12 Conclusions and final remarks

In this chapter, we introduce a new three-parameter model, called the modified Fréchet (MF) distribution, to extend the Fréchet distribution. Using the Lambert W function, we study some of its structural properties. We can generate random variates from the new distribution using this function. Some plots for the density and hazard rate functions are obtained. In addition, we also obtain explicit expressions for the quantile function, ordinary moments, generating function and Bonferroni and Lorenz curves for the new distribution. Moments of the order statistics also are investigated. Maximum likelihood estimates for complete samples are considered and we perform a Monte Carlo simulation in order to evaluate the behavior of these estimates in finite samples. We compare the performance of the new model with other extended Fréchet distributions including the exponentiated Weibull distribution by using the classical goodness-of-fit statistics and Q-Q plots. The results confirm that the new distribution can be appropriated for applications to real data sets.



(a) Fréchet vs MF

(b) EF vs MF

(c) MOF vs MF

(d) EW vs MF

Figure 3.7: Q-Q plots for the Fréchet, EF, MOF and EW distributions (solid circles) vs Q-Q plot for the MF distribution (open circles) for the plasma ferritin data.

Chapter 4

The beta Marshall-Olkin Lomax distribution

Resumo

A composição de distribuições é o método mais comum para obter famílias de distribuições mais flexíveis. Considerando a família beta Marshall-Olkin generalizada, introduzimos uma nova distribuição de quatro parâmetros, denominada beta Marshall-Olkin Lomax, para aplicações de tempo de vida. Obtemos algumas de suas propriedades com base nas distribuições Lomax e exp-Lomax. Realizamos um estudo de simulação para ilustrar o desempenho das estimativas de máxima verossimilhança. Uma aplicação para dados não censurados é considerada e usamos estatísticas de bondade de ajuste para avaliar a flexibilidade da nova distribuição, provando empiricamente que este modelo pode ser apropriado para aplicações de tempo de vida.

Palavras-chave: Análise de tempo de vida, distribuições generalizadas, distribuição Lomax, família beta-G, família Marshall-Olkin estendida.

Abstract

Compounding distributions is the most common method in lifetime analysis to obtain more flexible families of distributions. Based on the beta Marshall-Olkin generated family, we present a new four-parameter distribution, so-called the beta Marshall-Olkin Lomax, for lifetime applications. We obtain some of its properties from those of well-established distributions. We provide a simulation study to illustrate the performance of the maximum likelihood estimates. An application to uncensored data is carried out and we use some goodness-of-fit statistics to study the flexibility of the new distribution, proving empirically that this model can be appropriate for lifetime applications.

Keywords: Beta-G family, generalized distributions, lifetime analysis, Lomax distribution, Marshall-Olkin extended family.

4.1 Introduction

In applications involving lifetime models such as survival analysis, demography, reliability, actuarial study and others, the distributions with positive real supports play a fundamental role. Because of this, in recent years, there is a growing interest in constructing new distributions to model ageing phenomena (LAI *et al.*, 2006; LAI, 2013). The method that has received most attention by researchers to generate new models is that one by compounding existing distributions, usually referred as generalized G families of distributions (TAHIR; NADARAJAH, 2015). The principal reason for this is the ability of these generalized distributions to be more flexible than the parent distribution to provide better fits to skewed data and good control of the tails (PESCIM *et al.*, 2010). The second reason is the powerful computational and analytical facilities available in several software packages, which facilitate handling and computing complex mathematical expressions. Some of the generalized G families best known are: the Marshall-Olkin extended (MOE) family (MARSHALL; OLKIN, 1997), the exponentiated-generated (exp-G) families (GUPTA *et al.*, 1998; CORDEIRO *et al.*, 2013), the beta-generated (beta-G) family (EUGENE *et al.*, 2002), the Kumaraswamy-generated (Kw-G) family (CORDEIRO; DE CASTRO, 2011), the gamma-generated (gamma-G) families (ZOGRAFOS; BALAKRISHNAN, 2009; RISTIĆ; BALAKRISHNAN, 2012; NADARAJAH *et al.*, 2015) and the McDonald-generated (Mc-G) family (ALEXANDER *et al.*, 2012). A detailed compilation of these families is given by Tahir & Nadarajah (2015).

In this chapter, we adopt the beta Marshall-Olkin generated (BMO-G) family proposed by Alizadeh *et al.* (2015) to define the new *beta Marshall-Olkin Lomax* (BMOL) distribution obtained by taking the Lomax distribution (LOMAX, 1954) as the parent model. Given that the proposed distribution has positive real support, our objective is to define a wide flexible distribution for real lifetime applications.

The chapter unfolds as follows. In Section 4.2, we describe some preliminaries and introduce the BMOL distribution. In Section 4.3, we plot its density and hazard rate functions for some parameter values. In Section 4.4, we obtain an expansion for the BMOL density function as a linear combination of exp-Lomax and Lomax densities.

In Sections 4.5 - 4.10, we present explicit expressions for the quantile function (qf), moments, generating function, mean deviations, Bonferroni and Lorenz curves, Shannon entropy and order statistics. Section 4.11 is devoted to the maximum likelihood estimates (MLE's) for complete samples and, in Section 4.12, we carry out a simulation study to study the performance of these estimates. In Section 4.13, we consider an application of the BMOL distribution and compare it with others related distributions and with the Exponentiated Weibull (EW) distribution (MUDHOLKAR; SRIVASTAVA, 1993) based on some goodness-of-fit statistics. Finally, Section 4.14 concludes the chapter.

4.2 The new distribution

Marshall & Olkin (1997) pioneered a method of introducing an additional parameter to a distribution. If $G(x;\boldsymbol{\xi})$ is the parent distribution with parameter vector $\boldsymbol{\xi}$, then the cumulative distribution function (cdf) given by

$$F(x;c,\boldsymbol{\xi}) = \frac{G(x;\boldsymbol{\xi})}{c+(1-c)G(x;\boldsymbol{\xi})}, \quad c>0, \tag{4.1}$$

defines a new distribution with an extra shape parameter c. As commented by the authors, new parameters generally are introduced in order to expand families and add flexibility.

The cdf of the beta-G family is defined by

$$\begin{aligned} F(x;a,b,\boldsymbol{\xi}) &= \frac{B(G(x;\boldsymbol{\xi});a,b)}{B(a,b)} \\ &= \frac{1}{B(a,b)}\int_0^{G(x;\boldsymbol{\xi})} w^{a-1}(1-w)^{b-1}\,dw, \quad a,b>0, \end{aligned} \tag{4.2}$$

where $B(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}\,dw$ is the beta function and $B(z;a,b)=\int_0^z w^{a-1}(1-w)^{b-1}\,dw$ is the incomplete beta function. The generated distribution $F(x;a,b,\boldsymbol{\xi})$ has two extra shape parameters a and b. The beta G family was introduced by Eugene *et al.* (2002), who studied the properties of the beta-normal distribution. If the parent $G(x;\boldsymbol{\xi})$ in (4.2) is the Lomax distribution, we obtain the beta-Lomax distribution as defined by Rajab *et al.* (2013).

A generalization of these concepts, introduced by Alzaatreh *et al.* (2013), follows by considering the $T-X$ method. Let $R(x;\boldsymbol{\gamma})$ be a cdf with support $[d,e]$ and density $r(x;\boldsymbol{\gamma})$. For a given parent distribution $G(x;\boldsymbol{\xi})$, let $W(\cdot)$ be a function satisfying the following properties:

$$\begin{cases} W[G(x;\boldsymbol{\xi})] \in [d,e], \\ W[G(x;\boldsymbol{\xi})] \text{ is differentiable and monotonically non-decreasing,} \\ \lim_{x\to-\infty} W[G(x;\boldsymbol{\xi})] = d, \quad \lim_{x\to\infty} W[G(x;\boldsymbol{\xi})] = e. \end{cases}$$

Then, the cdf

$$F(x;\boldsymbol{\delta},\boldsymbol{\gamma},\boldsymbol{\xi}) = \int_d^{W[G(x;\boldsymbol{\xi})]} r(t;\boldsymbol{\gamma})\,dt \tag{4.3}$$

defines a new distribution, where the link function $W(\cdot) = W(\cdot;\boldsymbol{\delta})$ possibly depends on a parameter vector $\boldsymbol{\delta}$. We say that the distribution $R(x;\boldsymbol{\gamma})$ is 'transformed' by the 'transformer' $W[G(x;\boldsymbol{\xi})]$.

Following this idea, Alizadeh *et al.* (2015) introduced the BMO-G family by considering in (4.3) the function $W(z) = z/[c + (1-c)z]$, $c > 0$, and the beta distribution as the 'transformed' distribution $R(x;\boldsymbol{\gamma})$. Notice that, in this case, the 'transformer' $W[G(x;\boldsymbol{\xi})]$ is given by (4.1).

In this chapter, we study the BMOL distribution by considering the parent $G(x;\boldsymbol{\xi})$ in (4.3) as the Lomax distribution (LOMAX, 1954), which has cdf given by

$$G(x;\alpha,\lambda) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}, \ x \geq 0, \, \alpha > 0, \, \lambda > 0 \tag{4.4}$$

and probability density function (pdf)

$$g(x;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)}. \tag{4.5}$$

For the sake of simplicity, we will write sometimes the Lomax distribution with cdf $G(x)$ and pdf $g(x)$, respectively, without explicit mention to the parameters α and λ.

It is clear that a generalized G distribution has more parameters than the parent distribution. Generally, the use of four parameters should be sufficient for most practical purposes. In addition, notice that if $X \sim \text{Lomax}(\alpha,\lambda)$, then $X/\lambda \sim \text{Lomax}(\alpha,1)$ and, consequently, λ is just a scale parameter. Henceforth, we consider the BMOL distribution with only four parameters by taking, without loss of generality, $\lambda = 1$ in equations (4.4) and (4.5). Thus, if $\boldsymbol{\theta} = (a,b,c,\alpha)^\top$ is the parameter vector, we define the BMOL cdf by

$$F(x;\boldsymbol{\theta}) = \frac{B(W[G(x)];a,b)}{B(a,b)} = \frac{1}{B(a,b)}\int_0^{W[G(x)]} w^{a-1}(1-w)^{b-1}\,dw, \tag{4.6}$$

where $W[G(x)]$ is given by (4.1). From equations (4.1) and (4.4) (with $\lambda = 1$), we obtain

$$W[G(x)] = \frac{1-(1+x)^{\alpha}}{1-c-(1+x)^{\alpha}}. \tag{4.7}$$

The BMOL pdf follows from (4.6) as

$$f(x;\boldsymbol{\theta}) = \frac{1}{B(a,b)}\, g(x)\, w[G(x)]\, \{W[G(x)]\}^{a-1}\, \{1-W[G(x)]\}^{b-1}, \tag{4.8}$$

where $w(z) = W'(z) = c/\left[c+(1-c)z\right]^2$. Thus, we obtain the BMOL pdf from (4.4), (4.7) and (4.8) as

$$f(x;\boldsymbol{\theta}) = \frac{\alpha\, c^b\, (1+x)^{-b\alpha-1}\left[1-(1+x)^{-\alpha}\right]^{a-1}}{\left\{c+(1-c)\,\left[1-(1+x)^{-\alpha}\right]\right\}^{a+b}\, B(a,b)}. \tag{4.9}$$

Hereafter, a random variable X with pdf (4.9) will be denoted by $X \sim \mathrm{BMOL}(a,b,c,\alpha)$.

In lifetime analysis, a very useful function is the hazard rate function (hrf) $r(x)$. Therefore, the hrf of X is given by

$$r(x) = \frac{\alpha\, c^b\, (1+x)^{-b\alpha-1}\left[1-(1+x)^{-\alpha}\right]^{a-1}}{\left\{c+(1-c)\,\left[1-(1+x)^{-\alpha}\right]\right\}^{a+b}\, \left[B(a,b)-B(W[G(x)],a,b)\right]}. \tag{4.10}$$

A random variable X with pdf (4.9) is easily simulated as follows: if $U \sim \mathrm{Beta}(a,b)$, then

$$X = Q(U) = \left[\left(\frac{1-(1-c)\,U}{1-U}\right)^{1/\alpha} - 1\right] \sim \mathrm{BMOL}(a,b,c,\alpha).$$

For specific values of the parameters a, b and c, some known sub-models of the BMOL distribution are given in Table 4.1.

4.3 Shapes of the density and hazard rate functions

The shapes of the pdf (4.9) can be described analytically by examining the roots of the equation $f'(x) = 0$ and analyzing its limits in (4.9) when $x \to 0$ or $x \to \infty$. Cleary, since $f(x) \geq 0$ is integrable, then $\lim_{x\to\infty} f(x) = 0$. The behavior of $f(x)$ when $x \to 0$ is governed by the parameter a, which is inherited from the properties of the beta distribution. For $a \leq 1$, we have that $f(x)$ is convex and strictly decreasing.

Table 4.1: Some BMOL sub-models. Lo: Lomax, MOL: Marshall-Olkin Lomax, KwL: Kumaraswamy Lomax, BL: beta Lomax

a	b	c	Model	Reference
1	1	1	Lo(α,1)	Lomax (1954)
1	1	-	MOL(c,α,1)	Ghitany *et al.* (2007)
1	-	1	KwL(1,b,α,1)	Shams (2013)
-	-	1	BL(a,b,α,1) (with $\mu=0$)	Rajab *et al.* (2013)

For $a = 1$, $\lim_{x\to 0} f(x) = b\,\alpha/c$ and, for $a < 1$, $\lim_{x\to 0} f(x) = \infty$. For $a > 1$, $f(0) = 0$ and it is unimodal with mode at

$$x_0 = -1 + \left\{ \frac{A_{a,b,c,\alpha} + \left[A^2_{a,b,c,\alpha} - 4\,(c-1)\,(\alpha-1)\,(b\,\alpha+1)\right]^{1/2}}{2\,(b\,\alpha+1)} \right\}^{1/\alpha},$$

where $A_{a,b,c,\alpha} = 2 - c - \alpha + b\,\alpha + a\,c\,\alpha$. All parameters allow extensive control on the right tail, providing, when $a > 1$, more light or heavy tails, according to the parameters decrease or increase, respectively, and conversely when $a \leq 1$. Some plots in Figure 4.1 display possible shapes of the pdf for selected parameter values. These plots confirm the above analysis.

The corresponding hrf can have the classical shapes such as decreasing or unimodal, as shown in Figure 4.2. Therefore, the new distribution can be appropriate for different applications in lifetime analysis.

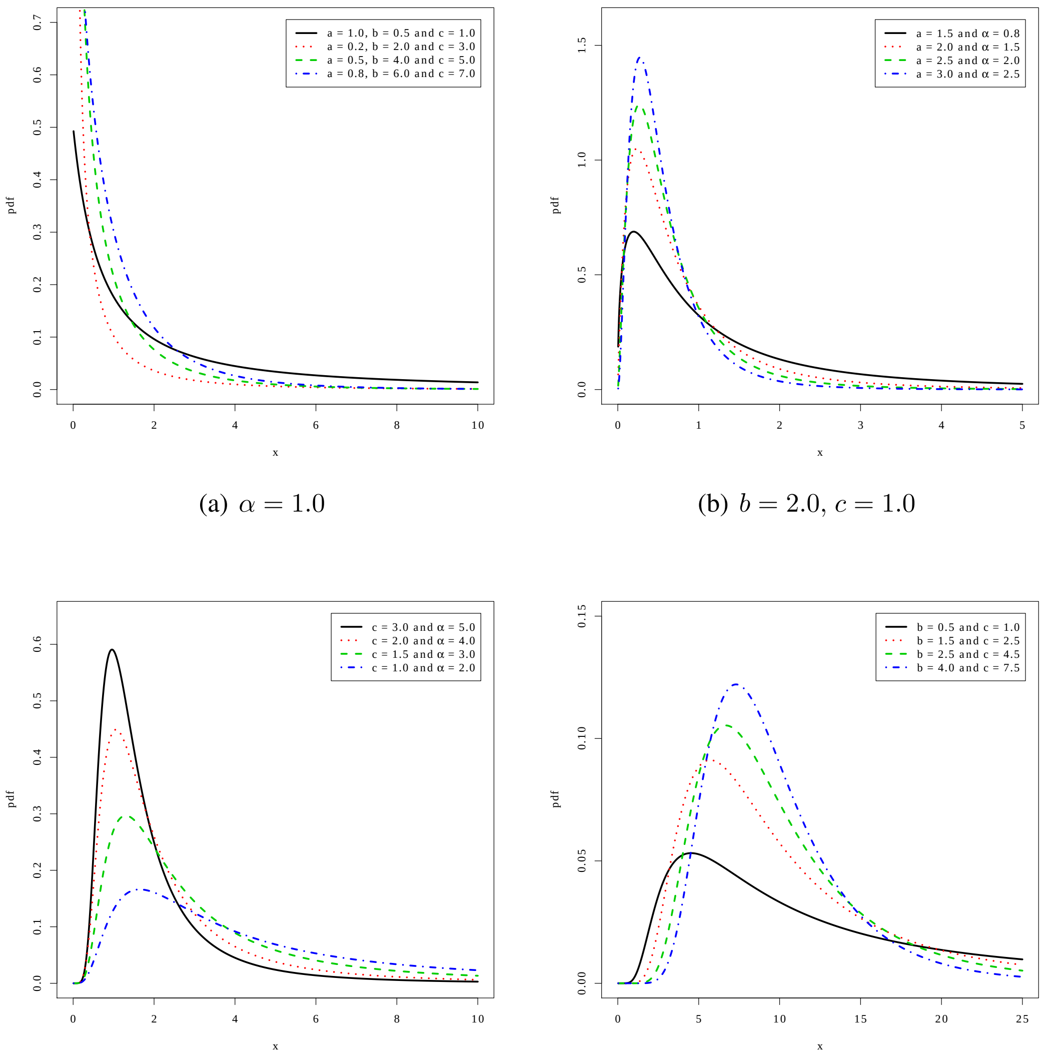
4.4 Useful representation

Using the generalized binomial expansion, Alizadeh *et al.* (2015) revealed that the cdf (4.6) admits the following power series

$$F(x) = \sum_{k=0}^{\infty} s_k\, G^k(x), \tag{4.11}$$

where $G(x)$ is the parent cdf (4.4) (with $\lambda = 1$) and, for $k \geq 0$,

$$s_k = \sum_{i,j=0}^{\infty} \sum_{l=k}^{\infty} \frac{(-1)^{i+l+k}(1-c)^i \binom{b-1}{i}\binom{-a-i}{j}\binom{a+i+j}{l}\binom{l}{k}}{c^{a+i+j}\,(a+i)\,B(a,b)}. \tag{4.12}$$



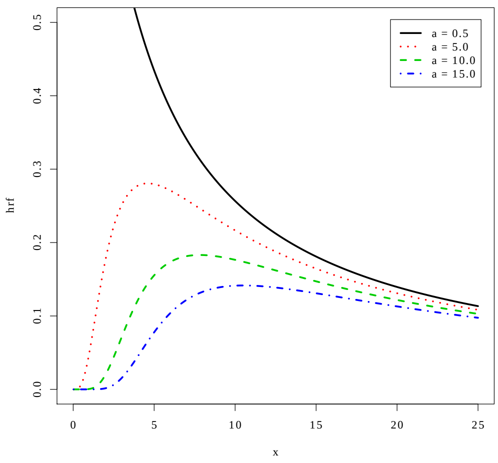
(a) $\alpha = 1.0$

(b) $b = 2.0$, $c = 1.0$

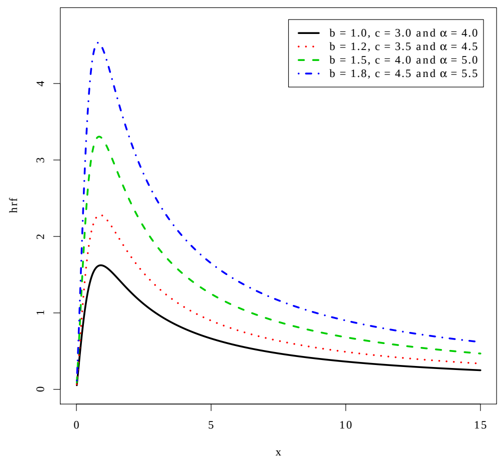
(c) $a = 7.0$, $b = 0.5$

(d) $a = 15.0$, $\alpha = 1.5$

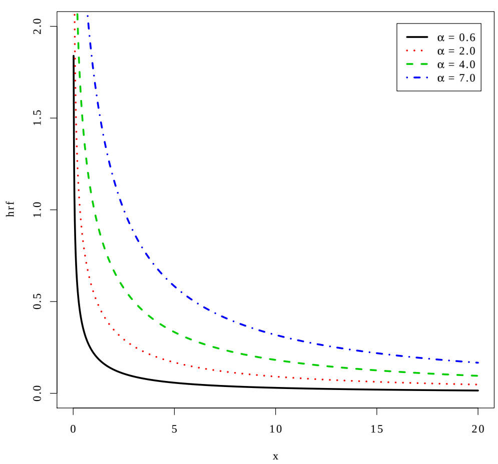
Figure 4.1: Plots of the BMOL pdf (4.9) for selected parameters.



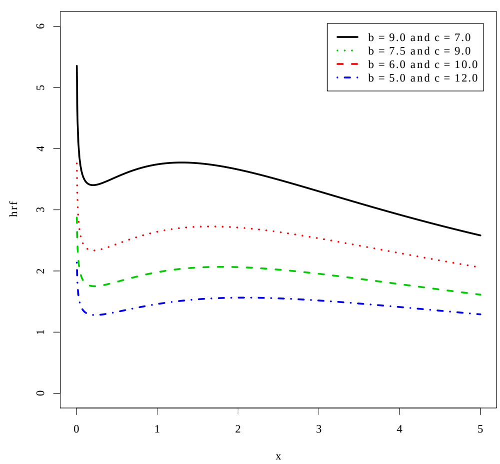
(a) $b = 2.0$, $c = 4.0$, $\alpha = 1.5$



(b) $a = 2.0$



(c) $a = 0.5$, $b = 0.5$, $c = 1.0$



(d) $a = 0.8$, $\alpha = 2.0$

Figure 4.2: Plots of the BMOL hrf (4.10) for selected parameters.

We note that (4.12) is valid only for $c > 1$, it does not converge for $c < 1$ and it is not applicable for $c = 1$. Differentiating (4.11) term by term, we obtain

$$f(x) = \sum_{k=0}^{\infty} s_{k+1}\, h_{k+1}(x), \tag{4.13}$$

where $h_{k+1}(x) = (k+1)\, g(x)\, G^{k}(x)$ denotes the exp-G density function with power parameter $k+1$. Therefore, from (4.13), several properties of the new model can be derived from those exp-G properties (MUDHOLKAR; SRIVASTAVA, 1993; MUDHOLKAR *et al.*, 1995; GUPTA *et al.*, 1998; GUPTA; KUNDU, 2001; CARRASCO *et al.*, 2008; CORDEIRO *et al.*, 2011).

It is possible to go a step further in (4.11). Using the binomial expansion in (4.11) gives

$$F(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}\, s_k\,(1+x)^{-j\alpha}.$$

By exchanging the indices j and k in the sums, we can write

$$F(x) = \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}(-1)^{j}\binom{k}{j}\, s_k\,(1+x)^{-j\alpha}. \tag{4.14}$$

Finally, differentiating (4.14) term by term, we obtain

$$f(x) = \sum_{j=0}^{\infty} p_j\, g(x;(j+1)\,\alpha,1), \tag{4.15}$$

where $g(x;(j+1)\,\alpha,1)$ is given in (4.5) and, for $j = 0,1,\ldots$,

$$p_j = \sum_{k=j+1}^{\infty}(-1)^{j}\binom{k}{j+1}\, s_k. \tag{4.16}$$

From equation (4.15), we note that $f(x)$ is given by a linear combination of Lomax densities. Therefore, several properties of the BMOL distribution can be obtained from those of the Lomax distribution (LOMAX, 1954).

4.5 Quantile function

Let $Q_{a,b}(z)$ denote the qf of the beta distribution with parameters a and b. Then, the qf of the BMOL distribution is given by

$$Q(z) = \left[\frac{1-(1-c)\,Q_{a,b}(z)}{1-Q_{a,b}(z)}\right]^{1/\alpha} - 1. \tag{4.17}$$

An expansion up to third order about $z = 0$ for the beta qf is given by

$$Q_{a,b}(z) = \sum_{i=1}^{3} q_i\, z^{i/a} + \mathcal{O}(z^{4/a}),$$

where $q_i = d_i\,[a\,B(a,b)]^{i/a}$, $i = 1,2,3$, with $d_1 = 1$,

$$d_2 = \frac{b-1}{a+1}, \quad d_3 = \frac{(b-1)\,(a^2+3ab-a+5b-4)}{2(a+1)^2(a+2)}.$$

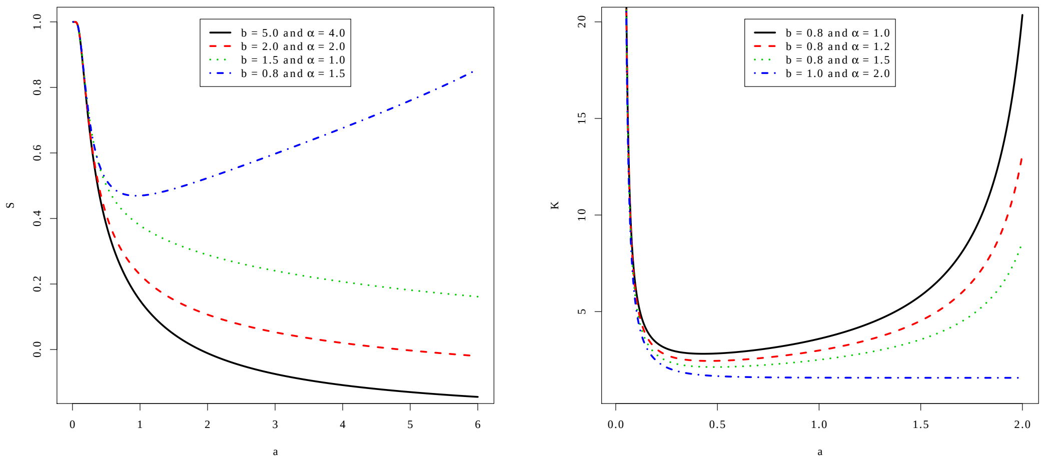
The skewness and kurtosis measures are determined by $\alpha_3 = \mu_3/\sigma^3$ and $\alpha_4 = \mu_4/\sigma^4$, respectively, where μ_j is the j-th central moment and σ is the standard deviation. For some generalized distributions obtained by the T–X method, as noted by Alzaatreh *et al.* (2013), it could be difficult to determine the third and fourth moments. Alternative measures for the skewness and kurtosis based on the qf are sometimes more appropriate. The measure of skewness S of Bowley and the measure of kurtosis K of Moors are defined by

$$S = \frac{Q(6/8)+Q(2/8)-2Q(4/8)}{Q(6/8)-Q(2/8)}, \tag{4.18}$$

$$K = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}. \tag{4.19}$$

These measures are more robust and they exist even for distributions without moments.

The plots in Figure 4.3 display the skewness (4.18) and kurtosis (4.19) as functions of the parameter a for some values of the parameters b, c and α. Note that, as pointed in Section 4.3, the BMOL pdf does not have mode when $a \leq 1$, which implies a greater skewness for these values of the parameter a, as ilustrated in Figure 4.3(a). Similary, note that the skewness increases when $b > 1$, obtaining negative values when $b,\alpha > 2$. In addition, note that the kurtosis decreases when the values of the parameters b and α increases, as ilustrated in Figures 4.3(b), 4.1(c) and 4.1(d).



(a) Skewness (b) Kurtosis

Figure 4.3: Plots of the skewness and kurtosis of the BMOL distribution for selected parameters ($c = 2.0$).

4.6 Moments

The moments of X can be expressed from the (r,k)-th probability weighted moment (PWM) of a random variable Y with parent cdf $G(x)$, which is defined, for $r,k = 0,1,\ldots$, by

$$\omega_{r,k} = \mathbb{E}\big[Y^r\, G^k(Y)\big] = \int_0^\infty y^r\, G^k(y)\, g(y)\, dy.$$

Setting $u = G(y)$, we obtain

$$\omega_{r,k} = \int_0^1 Q_G^r(u)\, u^k\, du, \tag{4.20}$$

where $Q_G(u)$ is the qf of $G(x)$.

The r-th ordinary moment of X, with $r \in \mathbb{N}$, follows from (4.13), for $c > 1$, as

$$\mu_r' = \sum_{k=0}^{\infty} \int_0^\infty x^r s_{k+1}\, h_{k+1}(x)\, dx,$$

where it is possible to exchange the infinite sum and the integral using the dominated convergence theorem. By using (4.20) and $h_{k+1}(x) = (k+1)\, g(x)\, G^k(x)$, we obtain

$$\mu'_r = \sum_{k=0}^{\infty} (k+1)\, s_{k+1} \int_0^1 Q_G^r(u)\, u^k\, du = \sum_{k=0}^{\infty} (k+1)\, s_{k+1}\, \omega_{r,k}. \tag{4.21}$$

Equation (4.21) reveals that the moments of the BMOL distribution can be expressed as an infinite weighted sum of the parent PWMs.

If $G(x)$ is the Lomax cdf (with $\lambda = 1$), we obtain, using the binomial expansion,

$$Q_G^r(z) = \left[\frac{1}{(1-z)^{1/\alpha}} - 1\right]^r = \sum_{j=0}^{r} \binom{r}{j} \frac{(-1)^{r+j}}{(1-z)^{j/\alpha}}$$

and therefore, from equation (4.20),

$$\omega_{r,k} = \sum_{j=0}^{r} (-1)^{r+j} \binom{r}{j} \int_0^1 \frac{u^k}{(1-u)^{j/\alpha}}\, du. \tag{4.22}$$

As a result, from (4.21) and (4.22), we obtain that $\mu'_r < \infty$ for $r < \alpha$ and $\mu'_r = \infty$ for $0 < \alpha \le r$, a condition that also holds for the Lomax distribution.

We can express the r-th ordinary moment of X as a linear combination of the r-th ordinary moments of Lomax random variables. In fact, for $j = 0,1,\ldots$, let $\alpha_j = (j+1)\,\alpha$. By applying the dominated convergence theorem and using equation (4.15), we can write, for $c > 1$,

$$\mu'_r = \mathbb{E}(X^r) = \sum_{j=0}^{\infty} p_j \int_0^{\infty} x^r g(x;\alpha_j,1)\, dx = \sum_{j=0}^{\infty} p_j\, \mathbb{E}(Y_j^r),$$

where $Y_j \sim \text{Lomax}(\alpha_j,1)$.

From the equality $\mathbb{E}(Y_j^r) = \alpha_j\, \Gamma(r+1)\Gamma(\alpha_j - r)/\Gamma(\alpha_j+1)$ (LEMONTE; CORDEIRO, 2013), we obtain

$$\mu'_r = \Gamma(r+1) \sum_{j=0}^{\infty} p_j\, \alpha_j \frac{\Gamma(\alpha_j - r)}{\Gamma(\alpha_j + 1)}, \quad r < \alpha. \tag{4.23}$$

Equations (4.21) and (4.23) are the main results of this section. However, the moments of X can be determined from (4.23) more easily than from (4.21).

4.7 Generating function

A formula for the moment generating function (mgf) $M(t) = \mathbb{E}(\mathrm{e}^{\mathrm{tX}})$ of X follows from (4.13) as

$$M(t) = \sum_{k=0}^{\infty}(k+1)\, s_{k+1}\, \rho_k(t), \tag{4.24}$$

where

$$\rho_k(t) = \int_0^1 \exp\left[t\, Q_G(u)\right]\, u^k\, du = \int_0^1 \exp\left\{-t\left[1-(1-u)^{-1/\alpha}\right]\right\} u^k\, du.$$

We can obtain an expansion for $\rho_k(t)$, $t \geq 0$, using the gamma and the upper incomplete gamma functions, which are defined, respectively, as

$$\Gamma(\upsilon) = \int_0^{\infty} x^{\upsilon-1}\, \mathrm{e}^{-x}\, dx, \quad \Gamma(\upsilon{,}z) = \int_z^{\infty} x^{\upsilon-1}\, \mathrm{e}^{-x}\, dx, \quad \upsilon \in \mathbb{R},\, z \geq 0. \tag{4.25}$$

In fact, setting $w = 1-(1-u)^{-1/\alpha}$, we have $du = -\alpha\,(1-w)^{-\alpha-1}\, dw$ and, therefore,

$$\rho_k(t) = -\alpha \int_0^1 \mathrm{e}^{-tw}\left[1-(1-w)^{-\alpha}\right]^k\, (1-w)^{-\alpha-1}\, dw.$$

Using the binomial expansion, we obtain

$$\left[1-(1-w)^{-\alpha}\right]^k = \sum_{j=0}^{k}(-1)^j \binom{k}{j}\, (1-w)^{-j\alpha},$$

which leads to

$$\rho_k(t) = -\alpha \sum_{j=0}^{k}(-1)^j \binom{k}{j} \int_0^1 \mathrm{e}^{-tw}\, (1-w)^{-\alpha_j-1}\, dw, \tag{4.26}$$

where $\alpha_j = (j+1)\,\alpha$. We have $w \in (0{,}1)$ from (4.26) and, therefore, we can expand $(1-w)^{-\alpha_j-1}$ as

$$(1-w)^{-\alpha_j-1} = \sum_{r=0}^{\infty}(-1)^r \binom{-\alpha_j-1}{r}\, w^r,$$

where $\binom{-\alpha_j-1}{r} = (-1)^r(\alpha_j+1)(\alpha_j+2)\ldots(\alpha_j+r)/r!$. Thus, from (4.25) and (4.26), we obtain

$$\rho_k(t) = \alpha \sum_{j=0}^{k} \sum_{r=0}^{\infty} (-1)^{r+j-1} \binom{k}{j} \binom{-\alpha_j - 1}{r} t^{-r-1} \left[\Gamma(r+1) - \Gamma(r+1,t)\right]. \quad (4.27)$$

Equations (4.24) and (4.27) are the main results of this section.

4.8 Mean deviations and Bonferroni and Lorenz curves

As before, for $j = 0,1,\dots$, let $Y_j \sim \mathrm{Lomax}(\alpha_j,1)$. The mean deviations of X about the mean, $\delta_1 = \mathbb{E}|X - \mu_1'|$, and about the median, $\delta_2 = \mathbb{E}|X - M|$, can be expressed as

$$\delta_1 = 2\,\mu_1'\,F(\mu_1') - 2\,m_X^{(1)}(\mu_1') \;\text{ and }\; \delta_2 = \mu_1' - 2\,m_X^{(1)}(M),$$

where μ_1' is the first ordinary moment of X given by (4.23), $m_X^{(1)}(z) = \int_0^z x f(x)\,dx$ denotes the first incomplete moment of X, $M = Q(0.5)$ is the median of X and $Q(\cdot)$ is given by (4.17). The mean deviations δ_1 and δ_2 are used frequently as dispersion measures.

Using (4.15), we can write

$$m_X^{(1)}(z) = \sum_{j=0}^{\infty} p_j \int_0^z x\, g(x;\alpha_j,1)\,dx = \sum_{j=0}^{\infty} p_j\, m_{Y_j}^{(1)}(z), \quad (4.28)$$

where $m_{Y_j}^{(1)} = \int_0^z x\, g(x;\alpha_j,1)\,dx$ denotes the first incomplete moment of Y_j and p_j is given by (4.16). For computing δ_1 and δ_2, we use (4.6), (4.23) and (4.28).

The incomplete moments can be applied to obtain the Bonferroni and Lorenz curves, which are useful in several areas. The Bonferroni and Lorenz curves are defined, respectively, by

$$B(\pi) = \frac{m_X^{(1)}(q)}{\pi \mu_1'} \;\text{ and }\; L(\pi) = \frac{m_X^{(1)}(q)}{\mu_1'},$$

where $q = Q(\pi)$ is evaluated from (4.17) for $0 < \pi < 1$.

4.9 Entropy

Entropy is a measure of disorder or uncertainty. Two variants of entropy are generally used, the Shannon and Rényi entropies. The latter is a generalization of the first.

The Shannon entropy of a random variable X is defined by

$$\eta_X = \mathbb{E}_X\{-\log[f(X)]\} = -\int_{supp(f)} \log[f(x)]\,dF(x),$$

where $F(x)$ is the cdf of X and $supp(f)$ indicates the support of the density function $f(x)$.

Considering that $W[G(x)]$ is an absolutely continuous distribution with density $g(x)\,w[G(x)]$, where $G(x)$ is the parent distribution and $w(z) = W'(z)$ (see Section 4.2), we have

$$-\mathbb{E}_X\{\log(W[G(X)])\} = \xi(a,b),$$

$$-\mathbb{E}_X\{1 - \log(W[G(X)])\} = \xi(b,a),$$

and

$$\mathbb{E}_X\{\log(w[G(X)])\} + \mathbb{E}_X\{\log[g(X)]\} - \mathbb{E}_U\{\log[w(U)]\} - \mathbb{E}_U\{\log(g[Q_G(U)])\} = 0,$$

where $\xi(a,b) = -\frac{\partial}{\partial a}\log[B(a,b)] = \psi(a+b) - \psi(a)$, $\psi(\cdot)$ denotes the digamma function and $U \sim \text{Beta}(a,b)$.

From the equalities $w(z) = c/[c + (1-c)z]^2$ and $g(Q_G(u)) = \alpha(1-u)^{(\alpha+1)/\alpha}$, we obtain

$$\mathbb{E}_U\{\log[w(U)]\} = \log c - 2\,\mathbb{E}_U\{\log[c + (1-c)\,U]\},$$
$$\mathbb{E}_U\{\log(g[Q_G(U)])\} = \log\alpha + \frac{\alpha+1}{\alpha}\,\mathbb{E}_U[\log(1-U)].$$

Further, we have

$$\begin{aligned}\mathbb{E}_U\{\log(1-U)\} &= \frac{1}{B(a,b)}\int_0^1 \log(1-u)\,u^{a-1}\,(1-u)^{b-1}\,du\\ &= -\,\xi(b,a),\\ \mathbb{E}_U\{\log[c + (1-c)\,U]\} &= \frac{1}{B(a,b)}\int_0^1 \log[c + (1-c)u]\,u^{a-1}\,(1-u)^{b-1}\,du\\ &= \log c - I_{a,b,c}\;{}_3F_2(1,1,1+a;2,1+a+b;\frac{c-1}{c}),\end{aligned}$$

where $I_{a,b,c} = \frac{(c-1)\,B(1+a,b)}{c\,B(a,b)}$ and ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is the generalized hypergeometric function. Thus, we can write

$$\eta_X = \log[B(a,b)] + (a-1)\,\xi(a,b) + \left(b - 1 + \frac{\alpha+1}{\alpha}\right)\xi(b,a) + \log c - \log\alpha$$
$$- 2\, I_{a,b,c}\ {}_3F_2\left(1,1,1+a; 2,1+a+b; \frac{c-1}{c}\right).$$

The Shannon entropy is relevant because it is related to other notions of entropy in various areas such as probability theory, computer sciences, dynamical systems and statistical physics.

4.10 Order statistics

Let $X_1,\dots,X_n$ be a random sample of size n from a distribution $F(x)$. Then, the pdf of the m-th order statistic, $X_{(m)}$, is given by (SEVERINI, 2005, p. 218)

$$f_{(m)}(x) = K\, F^{m-1}(x)\,[1 - F(x)]^{n-m} f(x),$$

where $K = n!/[(m-1)!\,(n-m)!]$. For $1 \le m < n$, we obtain

$$f_{(m)}(x) = K\, f(x) \sum_{j=0}^{n-m} (-1)^j \binom{n-m}{j} F^{m+j-1}(x).$$

Based on (4.11) and (4.12) and using an expansion for power series raised to powers (GRADSHTEYN; RYZHIK, 2007, p. 17), we have, for $c > 1$,

$$F^{m+j-1}(x) = \left(\sum_{k=0}^{\infty} s_k\, G^k(x)\right)^{m+j-1} = \sum_{k=0}^{\infty} v_k\, G^k(x),$$

where $G(x)$ is the parent distribution given in (4.4) (with $\lambda = 1$), $v_0 = s_0^{m+j-1}$ and, for $i \ge 1$,

$$v_i = \frac{1}{i\, s_0} \sum_{l=1}^{i} [(m+j)l - i]\, r_l\, s_{i-l}.$$

Therefore, we obtain

$$f_{(m)}(x) = K f(x) \sum_{j=0}^{n-m} \sum_{k=0}^{\infty} (-1)^j \binom{n-m}{j} v_k\, G^k(x),$$

where the density $f(x)$ is given in (4.9). For $n = m$, we have

$$f_{(n)}(x) = K\, F^{n-1}(x) f(x).$$

From (4.11) and (4.12), result

$$F^{n-1}(x) = \left(\sum_{k=0}^{\infty} s_k\, G^k(x) \right)^{n-1} = \sum_{k=0}^{\infty} \tilde{v}_k\, G^k(x),$$

where $\tilde{v}_0 = s_0^{n-1}$ and, for $i \geq 1$,

$$\tilde{v}_i = \frac{1}{i\, s_0} \sum_{l=1}^{i} (nl - i)\, s_l\, \tilde{v}_{i-l}.$$

Thus, we obtain

$$f_{(n)}(x) = K\, f(x) \sum_{k=0}^{\infty} \tilde{v}_k\, G^k(x).$$

Alizadeh *et al.* (2015) proposed other expansion for $f_{(m)}(x)$, $1 \leq m < n$, given by

$$f_{(m)}(x) = \sum_{r,k=0}^{\infty} p_{r,k}\, h_{r+k+1}(x), \tag{4.29}$$

where $h_{r+k+1}(x)$ denotes the exp-G density function with parameter $r + k + 1$,

$$p_{r,k} = \frac{n!\,(r+1)\,(m-1)!\, s_{r+1}}{r+k+1} \sum_{j=0}^{n-m} \frac{(-1)^j\, v_{j,k}}{(n-i-j)!\, j!},$$

s_r is given in (4.12) for $c > 1$ and $v_{j,k}$ is determined recursively as $v_{j,0} = s_0^{j+m-1}$ and, for $k \geq 1$,

$$v_{j,k} = \frac{1}{k\, s_0} \sum_{l=1}^{k} [(j+m)l - k]\, s_l\, v_{j,k-l}.$$

Equation (4.29) reveals that, for $1 \leq m < n$, the density function $f_{(m)}(x)$ of the m-th order statistic $X_{(m)}$ can be expressed as a linear mixture of exp-G densities. Therefore, some structural properties of $X_{(m)}$ can be obtained from those of the exp-G distribution.

4.11 Maximum Likelihood Estimation

Several approaches for parameter estimation were proposed in the statistical literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties for constructing confidence intervals. In this section, we estimate the parameters of the BMOL distribution by maximum likelihood for complete data sets. Let $\boldsymbol{x} = (x_1, \ldots, x_n)^\top$ be a sample of size n from $X \sim \mathrm{BMOL}(a,b,c,\alpha)$ and $\boldsymbol{\theta} = (a,b,c,\alpha)^\top$ the parameter vector. The log-likelihood for $\boldsymbol{\theta}$, denoted by $\ell_f(\boldsymbol{\theta};\boldsymbol{x})$, is given by

$$\begin{aligned}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) =& -n\log[B(a,b)] + (a-1)\sum_{i=1}^{n}\log\{W[G(x_i)]\}\\ &+ (b-1)\sum_{i=1}^{n}\log\{1-W[G(x_i)]\} + \sum_{i=1}^{n}\log\{w[G(x_i)]\} + \ell_g(\alpha;\boldsymbol{x}),\end{aligned}$$

where $\ell_g(\alpha;\boldsymbol{x}) = \sum_{i=1}^{n}\log[g(x_i)]$ is the log-likelihood for the Lomax parameters (with $\lambda = 1$). From (4.4) and (4.7), we can write

$$\log\{W[G(x_i)]\} = \log\left[\frac{1-(1+x_i)^\alpha}{1-c-(1+x_i)^\alpha}\right],$$

$$\log\{1-W[G(x_i)]\} = \log\left[\frac{-c}{1-c-(1+x_i)^\alpha}\right]$$

and

$$\log\{w[G(x_i)]\} = \log\left\{\frac{c\,(1+x_i)^{2\alpha}}{[1-c-(1+x_i)^\alpha]^2}\right\}.$$

Then,

$$\begin{aligned}\ell_f(\boldsymbol{\theta};\boldsymbol{x}) =& -n\log[B(a,b)] + (a-1)\sum_{i=1}^{n}\log\left[\frac{1-(1+x_i)^\alpha}{1-c-(1+x_i)^\alpha}\right]\\ &+ (b-1)\sum_{i=1}^{n}\log\left[\frac{-c}{1-c-(1+x_i)^\alpha}\right]\\ &+ \sum_{i=1}^{n}\log\left\{\frac{c\,(1+x_i)^{2\alpha}}{[1-c-(1+x_i)^\alpha]^2}\right\} + \ell_g(\alpha;\boldsymbol{x}).\end{aligned} \tag{4.30}$$

The MLE $\hat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ can be obtained by maximizing (4.30) directly by using a package software. Alternatively, we can obtain the components of the score vector $\boldsymbol{U}_{\boldsymbol{\theta}} =$

$(U_a,U_b,U_c,U_\alpha)^\top$ and set them to zero. They are given by

$$U_a = n[\psi(a+b) - \psi(a)] + \sum_{i=1}^{n} \log\left[\frac{1-(1+x_i)^\alpha}{1-c-(1+x_i)^\alpha}\right],$$

$$U_b = n[\psi(a+b) - \psi(b)] + \sum_{i=1}^{n} \log\left[\frac{-c}{1-c-(1+x_i)^\alpha}\right],$$

$$\begin{aligned} U_c = &\frac{1}{c}\sum_{i=1}^{n}\frac{1+c-(1+x_i)^\alpha}{1-c-(1+x_i)^\alpha} + \frac{(b-1)}{c}\sum_{i=1}^{n}\frac{1-(1+x_i)^\alpha}{1-c-(1+x_i)^\alpha} \\ &- (a-1)\sum_{i=1}^{n}\frac{(1+x_i)^{-\alpha}}{c+(1-c)\left[1-(1+x_i)^{-\alpha}\right]} \end{aligned}$$

and

$$\begin{aligned} U_\alpha = &\, n\log\alpha - (\alpha+1)\sum_{i=1}^{n}\log(1+x_i) - 2(c-1)\sum_{i=1}^{n}\frac{\log(1+x_i)}{1-c-(1+x_i)^\alpha} \\ &+ c(a-1)\sum_{i=1}^{n}\frac{(1+x_i)^\alpha\,\log(1+x_i)}{\left[1-(1+x_i)^\alpha\right]\left[1-c-(1+x_i)^\alpha\right]} \\ &+ (b-1)\sum_{i=1}^{n}\frac{(1+x_i)^\alpha\,\log(1+x_i)}{1-c-(1+x_i)^\alpha}. \end{aligned}$$

The MLE $\hat{\boldsymbol{\theta}}_n$ is obtained by solving the equations $U_a = U_b = U_c = U_\alpha = 0$ simultaneously. Because they can not be solved in closed-form, numerical iterative Newton-Raphson type algorithms can be applied.

Under general regularity conditions, we have $(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \overset{a}{\sim} N_4(\mathbf{0},K(\boldsymbol{\theta})^{-1})$, where $K(\boldsymbol{\theta})$ is the 4×4 expected information matrix and $\overset{a}{\sim}$ denotes asymptotic distribution. For n large, $K(\boldsymbol{\theta})$ can be approximated by the observed information matrix. This normal approximation for the MLE $\hat{\boldsymbol{\theta}}_n$ can be used for construing approximate confidence intervals and for testing hypotheses on the parameters a,b,c and α.

Suposse that the parameter vector is particioned as $\boldsymbol{\theta} = (\boldsymbol{\psi}_1^\top,\boldsymbol{\psi}_2^\top)^\top$, where $\dim(\boldsymbol{\psi}_1)+\dim(\boldsymbol{\psi}_2) = \dim(\boldsymbol{\theta})$. The likelihood ratio (LR) statistic for testing the null hypothesis $\mathcal{H}_0 : \boldsymbol{\psi}_1 = \boldsymbol{\psi}_1^{(0)}$ against the alternative hypothesis $\mathcal{H}_1 : \boldsymbol{\psi}_1 \neq \boldsymbol{\psi}_1^{(0)}$ is given by $\mathrm{LR}_n = 2\{\ell_f(\hat{\boldsymbol{\theta}}_n) - \ell_f(\tilde{\boldsymbol{\theta}}_n)\}$, where $\hat{\boldsymbol{\theta}}_n = (\hat{\boldsymbol{\psi}}_1^\top,\hat{\boldsymbol{\psi}}_2^\top)^\top$, $\tilde{\boldsymbol{\theta}}_n = (\boldsymbol{\psi}_1^{(0)^\top},\tilde{\boldsymbol{\psi}}_2^\top)^\top$, $\hat{\boldsymbol{\psi}}_i$ and $\tilde{\boldsymbol{\psi}}_i$ are the MLE's under the alternative and null hypotheses, respectively, and $\boldsymbol{\psi}_1^{(0)}$ is a specified parameter vector. Based on the first-order asymptotic theory, we know that $\mathrm{LR}_n \overset{a}{\sim} \chi_k^2$, where $k = \dim(\boldsymbol{\psi}_1)$. Thus, we can compute the maximum values of the unrestricted and restricted log-likelihoods to obtain LR statistics for testing some sub-models of the BMOL distribution (see Table 4.1).

4.12 Simulation study

In this section, we perform a Monte Carlo simulation experiment to evaluate the behavior of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{a}_n,\hat{b}_n,\hat{c}_n,\hat{\alpha}_n)$ in finite samples and estimate the relative bias and mean squared error (MSE) of the estimates for the sample sizes $n = 100, 200$ and 300. We consider $10{,}000$ Monte Carlo replications and use the BFGS method with analytical derivatives to maximize the log-likelihood function (4.30). We set the parameter values $a = 0.5$, $c = 0.25$ and vary b in the set $\{0.5, 0.75, 1.0\}$ and α in $\{0.5, 0.75\}$. All computations are performed using the `C` programming language and the `GNU` Scientific Library (version 2.1).

The results given in Table 4.2 reveal that, generally, the relative bias and MSE values decrease when n increases, which is to be expected since the MLE's are asymptotically unbiased. The minimum absolute values for the relative biases and MSEs are equal to 0.003. In counterpart, the maximum absolute values for the relative biases and MSEs are, respectively, 0.927 and 2.276. Further, it can be noted in Table 4.2 that the parameter c was underestimated in some cases (negative relative bias).

4.13 Application

In this section, the potentiality of the BMOL distribution is proved empirically by means of one lifetime application. We use an uncensored data set corresponding to 84 data on service times for failured windshields (MURTHY *et al.*, 2004, Table 16.11) and fit the BMOL distribution and its sub-models (see Table 4.1) to these data. All computations are done using the `R` software (version 3.0.2, `AdequacyModel` package). The descriptive statistics for this data set are given in Table 4.3.

For maximizing the log-likelihood function (4.30), we use the BFGS method with numerical derivatives. The MLE's are given in Table 4.4 (with standard errors in parentheses). For purposes of comparison, we compute some goodness-of-fit statistics: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Cramér-von Mises Criterion (W*) and Anderson-Darling Criterion (A*) (CHEN; BALAKRISHNAN, 1995). In general, the smaller the values of these statistics are, the better the fit is. We also include in the comparison the EW distribution (MUDHOLKAR; SRIVASTAVA, 1993), since it is a widely used

Table 4.2: Relative bias and MSE values of the MLE $\hat{\boldsymbol{\theta}}_n = (\hat{a}_n,\hat{b}_n,\hat{c}_n,\hat{\alpha}_n)$ for the BMOL model (with $a = 0.5$ and $c = 0.25$)

			relative bias				MSE			
b	α	n	$\hat{a}_n$	$\hat{b}_n$	$\hat{c}_n$	$\hat{\alpha}_n$	$\hat{a}_n$	$\hat{b}_n$	$\hat{c}_n$	$\hat{\alpha}_n$
0.5	0.5	100	0.112	0.166	−0.003	0.320	0.032	0.179	0.040	0.256
		200	0.053	0.116	−0.019	0.170	0.008	0.098	0.016	0.121
		300	0.034	0.082	−0.015	0.115	0.005	0.060	0.010	0.076
	0.75	100	0.113	0.191	0.029	0.297	0.046	0.296	0.105	0.546
		200	0.050	0.120	−0.013	0.161	0.008	0.099	0.016	0.261
		300	0.034	0.086	−0.013	0.110	0.004	0.064	0.010	0.167
0.75	0.5	100	0.093	0.092	0.080	0.598	0.035	0.382	0.170	0.602
		200	0.044	0.061	−0.018	0.336	0.006	0.166	0.015	0.259
		300	0.030	0.056	−0.025	0.221	0.004	0.123	0.009	0.148
	0.75	100	0.089	0.106	0.095	0.544	0.039	0.349	0.118	1.256
		200	0.043	0.064	−0.010	0.321	0.006	0.163	0.016	0.533
		300	0.029	0.059	−0.023	0.212	0.004	0.122	0.009	0.326
1.0	0.5	100	0.088	0.034	0.187	0.927	0.021	0.617	0.208	1.136
		200	0.047	0.012	0.006	0.560	0.005	0.273	0.022	0.522
		300	0.031	0.009	−0.008	0.389	0.003	0.192	0.012	0.295
	0.75	100	0.087	0.079	0.269	0.840	0.037	1.330	0.772	2.276
		200	0.046	0.019	0.014	0.522	0.005	0.256	0.024	1.066
		300	0.030	0.014	−0.005	0.367	0.003	0.188	0.012	0.613

lifetime model. Its cdf and pdf are given, respectively, by

$$H(x) = \left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\eta} \quad \text{and} \quad h(x) = \frac{\beta\eta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\left[1 - \mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right]^{\eta-1}\mathrm{e}^{-\left(\frac{x}{\alpha}\right)^{\beta}},$$

where $x \geq 0$ and α,β,$\eta > 0$.

The goodness-of-fit values for the fitted distributions are listed in Table 4.5.

Table 4.3: Descriptive statistics for the service times data

min.	1st quantile	median	mean	3rd quantile	max.
0.040	1.839	2.354	2.557	3.393	4.663

Based on the figures in Table 4.5, we note that the EW distribution presents the smaller values of the AIC, BIC and HQIC statistics. On the other hand, the BMOL distribution presents the smaller values of the W* and A* statistics. Since the BMOL and EW distributions are non-embedded models, a comparison between them is more appropriate by means of the W* and A* statistics. Also, note that the BMOL model presents the smaller value of the AIC statistic among all its sub-models and the smaller values of the BIC and HQIC statistics comparatively with the Lomax, BL and KwL distributions. Therefore, we can conclude that the BMOL distribution gives the best fit to the current data . If a minimum number of parameters is taken into account, the MOL or EW distributions can be chosen, since these also has less parameters.

To analyze how significant are the parameters of the BMOL distribution in modeling the current data, we use the LR statistic, as discussed in Section 4.11, for testing the BMOL model versus its sub-models listed in Table 4.1. The results are given in Table 4.6. Based on the figures in this table, we note that the rejection of the null hypotheses for the Lomax, MOL, BL and KwL models (at the 10% significance level) is significant. So, we have evidence of the potential need for including the parameters a,b and c to model the current data.

The plots of the estimated densities for the EW, MOL and BMOL distributions are displayed in Figure 4.4. Based on these plots, it is possible to assess the best overall fit of the BMOL distribution to the current data.

4.14 Conclusion and final remarks

In this chapter, we introduce a new four-parameter distribution, called the beta Marshall-Olkin Lomax (BMOL) distribution, as a member of the beta Marshall-Olkin generated (BMO-G) family (ALIZADEH *et al.*, 2015) when the parent model is the Lomax distribution (LOMAX, 1954) (with $\lambda = 1$). Some sub-models of the BMOL distribution are

Table 4.4: MLE's and standard errors for the service times data

Distribution	MLE					
	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$
Lo(α,1)	-	-	-	0.824 (0.090)	-	-
MOL(c,α,1)	-	-	441.875 (242.694)	4.957 (0.424)	-	-
BL(a,b,α,1)	6.664 (1.055)	38.687 (79.332)	-	0.133 (0.254)	-	-
KwL(a,b,α,1)	4.378 (0.517)	244.216 (213.820)	-	0.254 (0.083)	-	-
BMOL(a,b,c,α)	1.377 (0.356)	6.243 (5.526)	209.269 (143.799)	2.954 (0.627)	-	-
EW(α,β,η)	-	-	-	3.972 (0.136)	5.958 (0.255)	0.271 (0.036)

Table 4.5: Goodness-of-fit statistics for the service times data

Distribution	Statistic				
	AIC	BIC	HQIC	W*	A*
Lo($\hat{\alpha}$,1)	406.442	408.873	407.419	0.562	3.786
MOL($\hat{c}$,$\hat{\alpha}$,1)	266.987	271.849	268.942	0.068	0.650
BL($\hat{a}$,$\hat{b}$,$\hat{\alpha}$,1)	312.806	320.098	315.737	0.553	3.737
KwL($\hat{a}$,$\hat{b}$,$\hat{\alpha}$,1)	282.938	290.230	285.869	0.175	1.463
BMOL($\hat{a}$,$\hat{b}$,$\hat{c}$,$\hat{\alpha}$)	265.694	275.417	269.602	**0.048**	**0.487**
EW($\hat{\alpha}$,$\hat{\beta}$,$\hat{\eta}$)	**261.208**	**268.501**	**264.140**	0.129	0.831

Table 4.6: LR tests for the service times data

Models	Hypoteses	LR statistic	p-value
Lomax vs. BMOL	$\mathcal{H}_0$: $a = b = c = 1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	146.748	1.33×10^{-31}
MOL vs. BMOL	$\mathcal{H}_0$: $a = b = 1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	5.294	7.09×10^{-2}
BL vs. BMOL	$\mathcal{H}_0$: $c = 1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	49.112	2.42×10^{-12}
KwL vs. BMOL	$\mathcal{H}_0$: $a = c = 1$ vs. $\mathcal{H}_1$: $\mathcal{H}_0$ is false	19.244	6.63×10^{-5}

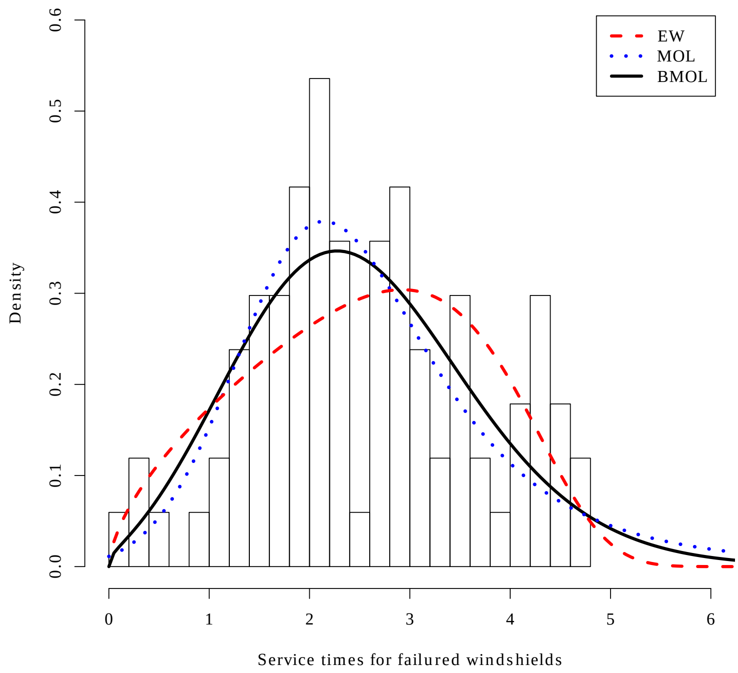


Figure 4.4: Fit comparison of the EW, MOL and BMOL estimated densities for the service times data.

presented. The new distribution has simple expressions for the cumulative and density functions. We study some of its mathematical and statistical properties. We demonstrate that the BMOL density can be expressed as linear combinations of Lomax and exponentiated-Lomax densities and therefore some of its structural properties can be obtained from those of these models. We present explicit expressions for the quantile function, moments, generating function, mean deviations, Shannon entropy and order statistics. We obtain the maximum likelihood estimates for complete samples and perform a Monte Carlo simulation in order to evaluate the behavior of these estimates in finite samples. We compare the performance of the new model with other related distributions including the exponentiated Weibull model using classical goodness-of-fit statistics. The results confirm that the BMOL distribution is very appropriate for lifetime applications.

Chapter 5

Final conclusions

Applying the parameter induction method, in this thesis are proposed new models primarily to be used in lifetime applications, although they have proved to be useful also for fit data of another nature. These new models extend or generalize classic lifetime models, adding flexibility to the parent distributions and improving the goodness-of-fit, which can be verified in the different applications presented throughout this thesis.

In Chapter 2, we introduce two new families of distributions, the supremum and infimum families, by inducing one additional shape parameter in the parent model G. These families have simple and elegant expressions and a physical interpretation in terms of functions of maximum and minimum the i.i.d. random variables. The probability density functions (pdf's) in both families have simple expressions in terms of pdf's the exponentiated-G (exp-G) distributions. Because of this, many properties of these families can be obtained from those of the exp-G distributions, such as expansions for the moments. In addition, it was proved that the supremum family can induce bathtub hazard rate, which has important implications in lifetime applications. These properties have motivated the introduction of the supremum and infimum families.

In Chapter 3, the modified Fréchet (MF) distribution is defined to extend the Fréchet distribution. Some mathematical quantities and properties of the new distribution are obtained by considering the Lambert W function, which arises in many mathematical and physical problems expressed in terms of logarithmics or exponential equations. Because the Lambert function has not yet been sufficiently explored in generalized probability distributions, the contribution of this chapter is valuable in showing how the analytical properties of this function can be used to obtain mathematical properties and quantities of generalized or extended models. Thus, we show how obtain an explicit expression for the quantile function of the MF distribution by using the Lambert W function and how obtain expansions for the ordinary moments, generating function and Bonferroni and Lorenz curves by using its analytical properties. The results on the

MF distribution was published in the journal *Communications in Statistics: Theory and Methods*, october 2016.

Chapter 4 introduces the beta Marshall-Olkin Lomax (BMOL) distribution by considering the Lomax distribution as the parent model in the generated beta Marshall-Olkin family (ALIZADEH *et al.*, 2015). It was demonstrated that the BMOL density can be expressed as linear combinations of Lomax and exp-Lomax densities. Thus, several of its structural properties can be obtained from the Lomax properties. Also, were presented explicit expresions for the quantile function, moments, generating function, mean deviations, Shannon entropy and order statistics. In this case, the new model have demostrated greater flexibility than its sub-models, providing more control in the tails.

At the end of each chapter, were considered applications to real data sets, which aim to assess the performance of the new distributions. For all cases, the results confirm that the proposed distributions are very appropiate for lifetime applications.

5.1 Future research

Next, we list some future topics to be investigated:

- It is possible to extend the G^{inf} family defined by equation (2.2) by inducing two additional shape parameters $a > 0$ and $b > 0$ as follows

$$F^{inf}(x;\boldsymbol{\xi},a,b) = H_a(x)\,\check{H}_b(x),$$

  where the basic family is obtained by taking $a = 1$. So, this extended family can be used as alternative method to existing ones, such as the generalized beta family (EUGENE *et al.*, 2002) or the generalized transmuted family (BOURGUIGNON *et al.*, 2016; NOFAL *et al.*, 2016) of distributions.

- Silva *et al.* (2010) defined the five-parameter beta modified Weibull distribution by considering the modified Weibull distribution introduced by Lai (2013) as the parent model in the generalized beta family. Following a similar approach, a future research line is to study the five-parameter *beta modified Fréchet* distribution. Because of the properties of the generalized beta family, the density function of the new distribution can be expressed as a linear mixture of MF distributions. Thus, several structural properties can be obtained from the latter. In addition, further mathematical quantities and properties of the new distribution can be obtained by considering the Lambert W function and follow the same approach given in Chapter 3.

References

AFIFY, A. Z.; HAMEDANI, G.; GHOSH, I.; MEAD, M. The transmuted Marshall-Olkin Fréchet distribution: Properties and applications. *International Journal of Statistics and Probability*, v. 4, p. 132–148, 2015.

ALEXANDER, C.; CORDEIRO, G. M.; ORTEGA, E. M. M.; SARABIA, J. M. Generalized beta-generated distributions. *Computational Statistics and Data Analysis*, v. 56, p. 1880–1897, 2012.

ALIZADEH, M.; CORDEIRO, G. M.; BRITO, E.; DEMÉTRIO, C. G. B. The beta Marshall–Olkin family of distributions. *Journal of Statistical Distributions and Applications*, Springer Berlin Heidelberg, v. 2, p. 1–18, 2015.

ALVES, I. F.; NEVES, C. Extreme value distributions. In: *International Encyclopedia of Statistical Science*. [S.l.]: Springer Berlin Heidelberg, 2011. p. 493–496.

ALZAATREH, A.; LEE, C.; FAMOYE, F. A new method for generating families of continuous distributions. *METRON*, Springer Milan, v. 71, p. 63–79, 2013.

BADER, M.; PRIEST, A. Statistical aspects of fibre and bundle strength in hybrid composites. *Progress in science and engineering of composites (Japan Society for Composite Materials)*, p. 1129–1136, 1982. Proceedings of ICCM IV, Tokyo, 1982.

BARAKAT, H. M.; ABDELKADER, Y. H. Computing the moments of order statistics from nonidentical random variables. *Statistical Methods and Applications*, Springer, v. 13, p. 15–26, 2004.

BARRETO SOUZA, W.; CORDEIRO, G. M.; SIMAS, A. B. Some results for beta Fréchet distribution. *Communications in Statistics–Theory and Methods*, Taylor & Francis, v. 40, p. 798–811, 2011.

BOURGUIGNON, M.; GHOSH, I.; CORDEIRO, G. M. General results for the transmuted family of distributions and new models. *Journal of Probability and Statistics*, Hindawi Publishing Corporation, n. to appear, 2016.

CARRASCO, J. M. F.; ORTEGA, E. M. M.; CORDEIRO, G. M. A generalized modified Weibull distribution for lifetime modeling. *Computational Statistics and Data Analysis*, v. 53, p. 450–462, 2008.

CHEN, G.; BALAKRISHNAN, N. A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, American Society for Quality, v. 27, p. 154–161, 1995.

CHICCOLI, C.; LORENZUTTA, S.; MAINO, G. Recent results for generalized exponential integrals. *Computers and Mathematics with Applications*, v. 19, p. 21 – 29, 1990.

CORDEIRO, G. M.; DE CASTRO, M. A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, v. 81, p. 883–898, 2011.

CORDEIRO, G. M.; ORTEGA, E. M. M.; CUNHA, D. C. C. da. The exponentiated generalized class of distributions. *Journal of Data Science*, v. 11, p. 1–27, 2013.

CORDEIRO, G. M.; ORTEGA, E. M. M.; SILVA, G. O. The exponentiated generalized gamma distribution with application to lifetime data. *Journal of Statistical Computation and Simulation*, v. 81, p. 827–842, 2011.

CORLESS, R. M.; GONNET, G. H.; HARE, D. E. G.; JEFFREY, D. J.; KNUTH, D. E. On the Lambert W function. *Advances in Computational mathematics*, Springer, v. 5, p. 329–359, 1996.

ELBATAL, I.; ASHA, G.; RAJA, A. V. Transmuted exponentiated Fréchet distribution: Properties and applications. *Journal of Statistics Applications and Probability*, v. 3, p. 379–394, 2014.

EUGENE, N.; LEE, C.; FAMOYE, F. Beta-normal distribution and its applications. *Communications in Statistics - Theory and Methods*, v. 31, p. 497–512, 2002.

GHITANY, M. E.; AL-AWADHI, F. A.; ALKHALFAN, L. A. Marshall–Olkin extended Lomax distribution and its application to censored data. *Communications in Statistics - Theory and Methods*, v. 36, p. 1855–1866, 2007.

GRADSHTEYN, I. S.; RYZHIK, I. M. *Table of integrals, series, and products*. Seventh. [S.l.]: Elsevier/Academic Press, Amsterdam, 2007.

GUPTA, R. C.; GUPTA, P. L.; GUPTA, R. D. Modeling failure time data by Lehmann alternatives. *Communications in Statistics - Theory and Methods*, v. 27, p. 887–904, 1998.

GUPTA, R. D.; KUNDU, D. Exponentiated exponential family: An alternative to gamma and Weibull distributions. *Biometrical Journal*, v. 43, p. 117–130, 2001.

HARLOW, D. G. Applications of the Fréchet distribution function. *International Journal of Materials and Product Technology*, Inderscience Publishers, v. 17, p. 482–495, 2002.

JODRÁ, P. Computer generation of random variables with Lindley or Poisson–Lindley distribution via the Lambert W function. *Mathematics and Computers in Simulation*, Elsevier, v. 81, p. 851–859, 2010.

JØRGENSEN, B. *Statistical properties of the generalized inverse Gaussian distribution*. [S.l.]: Springer-Verlag, 1982. (Lecture notes in statistics).

KOTZ, S.; NADARAJAH, S. *Extreme value distributions: theory and applications*. [S.l.]: World Scientific, 2000.

KRISHNA, E.; JOSE, K. K.; ALICE, T.; RISTIĆ, M. M. The Marshall-Olkin Fréchet distribution. *Communications in Statistics-Theory and Methods*, Taylor & Francis, v. 42, p. 4091–4107, 2013.

KULLBACK, S. *Information theory and statistics*. [S.l.]: Courier Corporation, 1997.

KUNDU, D.; RAQAB, M. Z. Estimation of $R = P(Y < X)$ for three-parameter weibull distribution. *Statistics & Probability Letters*, Elsevier, v. 79, p. 1839–1846, 2009.

LAI, C. D. Constructions and applications of lifetime distributions. *Applied Stochastic Models in Business and Industry*, v. 29, p. 127–140, 2013.

LAI, C. D.; XIE, M.; BARLOW, R. E. *Stochastic Ageing and Dependence for Reliability*. Secaucus, NJ, USA: Springer–Verlag New York, Inc., 2006.

LAI, C. D.; XIE, M.; MURTHY, D. N. P. A modified Weibull distribution. *IEEE Transactions on reliability*, IEEE, v. 52, p. 33–37, 2003.

LEMONTE, A. J.; CORDEIRO, G. M. An extended Lomax distribution. *Statistics*, v. 47, p. 800–816, 2013.

LOMAX. Business failures: Another example of the analysis of failure data. *Journal of the American Statistical Association*, Taylor & Francis, Ltd., v. 49, p. 847–852, 1954.

MAHMOUD, M. R.; MANDOUH, R. M. On the transmuted Fréchet distribution. *Journal of Applied Sciences Research*, v. 9, p. 5553–5561, 2013.

MARSHALL, A. W.; OLKIN, I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, v. 84, p. 641–652, 1997.

MARSHALL, A. W.; OLKIN, I. *Life Distributions: Structure of Nonparametric, Semiparametric, and Parametric Families*. [S.l.]: Springer New York, 2007. (Springer Series in Statistics).

MEAD, M. E. A. A note on Kumaraswamy Fréchet distribution. *Australian Journal of Basic and Applied Sciences*, v. 8, p. 294–300, 2014.

MUDHOLKAR, G. S.; SRIVASTAVA, D. K. Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, v. 42, p. 299–302, 1993.

MUDHOLKAR, G. S.; SRIVASTAVA, D. K.; FREIMER, M. The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics*, Taylor & Francis, Ltd., v. 37, p. 436–445, 1995.

MURTHY, D. N. P.; XIE, M.; JIANG, R. *Weibull models*. [S.l.]: John Wiley & Sons, 2004.

NADARAJAH, S.; CORDEIRO, G. M.; ORTEGA, E. M. M. The Zografos–Balakrishnan-G family of distributions: Mathematical properties and applications. *Communications in Statistics - Theory and Methods*, v. 44, p. 186–215, 2015.

NADARAJAH, S.; GUPTA, A. K. The beta Fréchet distribution. *Far East Journal of Theoretical Statistics*, v. 14, p. 15–24, 2004.

NADARAJAH, S.; KOTZ, S. The exponentiated Fréchet distribution. *Interstat Electronic Journal*, Citeseer, v. 31, p. 1–7, 2001.

NOFAL, Z. M.; AFIFY, A. Z.; YOUSOF, H. M.; CORDEIRO, G. M. The generalized transmuted-G family of distributions. *Communications in Statistics-Theory and Methods*, Taylor & Francis, n. just-accepted, 2016.

PESARAN, M. H.; ULLOA, M. R. D. Non-nested hypotheses. *The New Palgrave Dictionary of Economics, Second Edition. Palgrave Macmillan*, Citeseer, 2008.

PESCIM, R. R.; DEMÉTRIO, C. G. B.; CORDEIRO, G. M.; ORTEGA, E. M. M.; URBANO, M. R. The beta generalized half-normal distribution. *Computational Statistics and Data Analysis*, v. 54, p. 945–957, 2010.

RAJAB, M.; ALEEM, M.; NAWAZ, T.; DANIYAL, M. On five parameter beta Lomax distribution. *Journal of Statistics*, v. 20, p. 102–118, 2013.

RISTIĆ, M. M.; BALAKRISHNAN, N. The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*, v. 82, p. 1191–1206, 2012.

SEVERINI, T. *Elements of Distribution Theory*. [S.l.]: Cambridge University Press, 2005. (Cambridge Series in Statistical and Probabilistic Mathematics).

SHAMS, T. M. The Kumaraswamy-Generalized Lomax distribution. *Middle–East Journal of Scientific Research*, v. 17, p. 641–646, 2013.

SHAW, W. T.; BUCKLEY, I. R. The alchemy of probability distributions: beyond gram-charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. 2009.

SILVA, G. O.; ORTEGA, E. M. M.; CORDEIRO, G. M. The beta modified Weibull distribution. *Lifetime Data Analysis*, Springer, v. 16, p. 409–430, 2010.

SILVA, R. V. da; ANDRADE, T. A. N. de; MACIEL, D.; CAMPOS, R. P. S.; CORDEIRO, G. M. A new lifetime model: The gamma extended Fréchet distribution. *Journal of Statistical Theory and Applications*, Atlantis Press, v. 12, p. 39–54, 2013.

TAHIR, M. H.; NADARAJAH, S. Parameter induction in continuous univariate distributions: Well-established G families. *Anais da Academia Brasileira de Ciências*, scielo, v. 87, p. 539–568, 2015.

VUONG, Q. H. Likelihood ratio tests for model selection and non-nested hypotheses. *Econometrica: Journal of the Econometric Society*, v. 57, p. 307–333, 1989.

WEISBERG, S. *Applied linear regression*. [S.l.]: John Wiley & Sons, 2005.

YOUSOF, H. M. *et al.* The transmuted exponentiated generalized-G family of distributions. *Pakistan Journal of Statistics and Operation Research*, v. 11, p. 441–464, 2015.

ZOGRAFOS, K.; BALAKRISHNAN, N. On families of beta- and generalized gamma-generated distributions and associated inference. *Statistical Methodology*, v. 6, p. 344–362, 2009.